Evaluate $\underset{x \to 0}{Lt} \dfrac{(1+x)^{\frac{1}{x}} - e}{x}$

we have the given limit

$$= \underset{x \to 0}{Lt} \frac{(1+x)^{\frac{1}{x}} - e}{x}$$

$$= \underset{x \to 0}{Lt} \frac{e^{\log (1+x)^{\frac{1}{x}}} - e}{x}$$

$$= \underset{x \to 0}{Lt} \frac{e^{\frac{1}{x} \log(1+x)} - e}{x}$$

$$= \underset{x \to 0}{Lt} \frac{e^{\frac{1}{x}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\cdots\right]} - e}{x}$$

$$= \underset{x \to 0}{Lt} \frac{e^{1 - \frac{x}{2} + \frac{x^2}{3} - \cdots\cdots} - e}{x}$$

$$= \underset{x \to 0}{Lt} \frac{e \times e^{\left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\cdots\right)} - e}{x}$$

$$= \underset{x \to 0}{Lt} \frac{e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\cdots\right) + \frac{1}{\lfloor 2}\left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\right)^2 + \cdots\cdots\right] - e}{x}$$

$$= \underset{x \to 0}{Lt} \frac{e\left[\cancel{1} + \left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\cdots\right) + \frac{1}{\lfloor 2} x^2\left(-\frac{1}{2} + \frac{x}{3} - \cdots\cdots\right)^2 + \cdots\cdots - \cancel{1}\right]}{x}$$

$$= \underset{x \to 0}{Lt} \frac{e \cdot \cancel{x}\left[\left(-\frac{1}{2} + \frac{x}{3} - \cdots\cdots\right) + \frac{x}{\lfloor 2}\left(-\frac{1}{2} + \frac{x}{3} - \cdots\cdots\right)^2 + \cdots\cdots\right]}{\cancel{x}}$$

$$= e\left[-\frac{1}{2} + 0\right] = \frac{-e}{2}$$

(4) Prove that $\underset{x \to 0}{Lt} \log_{\tan x}(\tan 2x) = 1$

Ans → The given limit

$$= \underset{x \to 0}{Lt} \log_{\tan x}(\tan 2x)$$

$$= \underset{x \to 0}{Lt} \frac{\log_e \tan 2x}{\log_e \tan x} \quad \left[\because \log_n^m = \frac{\log_e^m}{\log_e^n}\right] \left[\frac{\infty}{\infty}\right]$$

$$= \underset{x \to 0}{Lt} \frac{\frac{1}{\tan 2x} \times 2\sec^2 2x}{\frac{1}{\tan x} \times \sec^2 x} \quad \left[\text{using L-Hopital's rule}\right]$$

$$= \underset{x \to 0}{Lt} \frac{\frac{\cos 2x}{\sin 2x} \times 2 \times \frac{1}{\cos^2 2x}}{\frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x}}$$

$$= \underset{x \to 0}{Lt} \frac{2 \sin x \cos x}{\sin 2x \cdot \cos 2x}$$

$$= 2 \cdot \underset{x \to 0}{Lt} \frac{\sin 2x}{2 \sin 2x \cos 2x} = 2 \cdot \underset{x \to 0}{Lt} \frac{\sin 2x}{\sin 4x} \left[\frac{0}{0}\right]$$

$$= 2 \cdot \underset{x \to 0}{Lt} \frac{2 \cos 2x}{4 \cos 4x} \quad \text{[By L-Hopital's rule]}$$

$$= 1$$

(4) Prove that $\underset{x \to \frac{\pi}{4}}{Lt} (\tan x)^{\tan 2x} = \frac{1}{e}$

Ans, Let $y = \underset{x \to \frac{\pi}{4}}{Lt} (\tan x)^{\tan 2x}$ $[1^\infty]$

$$\log y = \underset{x \to \frac{\pi}{4}}{Lt} \log (\tan x)^{\tan 2x} = \underset{x \to \frac{\pi}{4}}{Lt} \tan 2x \log \tan x$$

$$= \underset{x \to \frac{\pi}{4}}{Lt} \frac{\log \tan x}{\cot 2x} \left[\frac{0}{0}\right]$$

$$= \underset{x \to \frac{\pi}{4}}{Lt} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{(-)2 \operatorname{cosec}^2 2x} \quad \text{[By L-Hopital's rule]}$$

$$= \underset{x \to \frac{\pi}{4}}{Lt} (-) \frac{\sin^2 2x}{2} \times \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x}$$

$$= \underset{x \to \frac{\pi}{4}}{Lt} (-) \frac{\sin^2 2x}{\sin 2x} = \underset{x \to \frac{\pi}{4}}{Lt} (-) \sin 2x$$

$$\log y = -1$$
$$\text{or } y = e^{-1} = \frac{1}{e}$$

Tangents & Normals

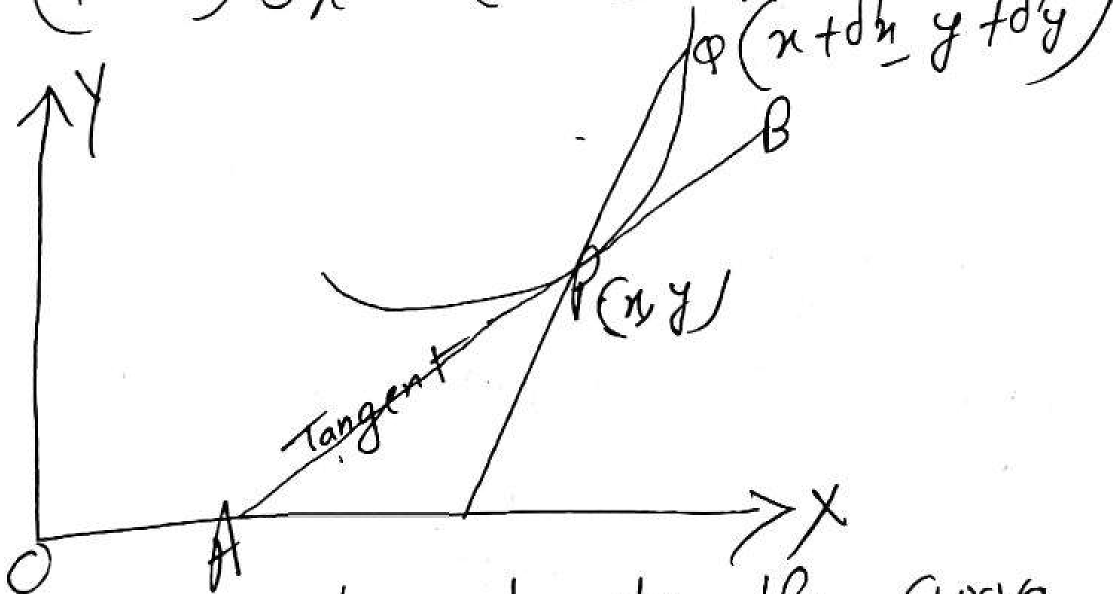
(Q) Prove that the equation to the tangent to the curve $y = f(x)$ at a point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

and tangent to the curve $f(x, y) = 0$ is

$$(X - x) f_x + (Y - y) f_y = 0$$

Ans →



Let AB be a tangent to the curve. AB touches the curve at a point $P(x, y)$. Let Q be the neighbouring point to P on the curve.

Let the coordinates of Q be $(x+\delta x, y+\delta y)$.

Let the Equation of the curve be $y = f(x)$.

8) Then $y + \delta y = f(x + \delta x)$ D.C.8)

as point Q $(x+\delta x, y+\delta y)$ lies on the curve.

$$\therefore \frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

Let (X, Y) be the current coordinates

Now equation of PQ is

$$\frac{Y - y}{X - x} = \frac{(y+\delta y) - y}{(x+\delta x) - x}$$

or $$\frac{Y - y}{X - x} = \frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

Now we assume that point Q tends to point P. when $Q \to P$ then $\delta x \to 0$ and the straight line PQ will supposed to be tangent to the curve at $P(x, y)$.

$$\therefore \frac{Y - y}{X - x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= f'(x) = \frac{dy}{dx}$$

Thus we find that the equation to the tangent to the curve $y = f(x)$ at point (x, y) is

$$\boxed{Y - y = \frac{dy}{dx}(X - x)} \quad \cdots (1)$$

 Again if the curve is of the form $f(x,y)=0$

$$\frac{dy}{dx} = \frac{(-)\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

or $$\frac{Y-y}{X-x} = \frac{(-)\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$ [using equation (1)]

or $$Y-y = (-)\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}(X-x)$$

or $$(X-x)\frac{\partial f}{\partial x} + (Y-y)\frac{\partial f}{\partial y} = 0$$

or $$\boxed{(X-x)f_x + (Y-y)f_y = 0}$$

this is the required equation to the tangent to the curve $f(x,y)=0$ at a point (x,y).

(*) Prove that the equation of the normal to the curve $y=f(x)$ at the point (x,y) is

$$(X-x) + (Y-y)\frac{dy}{dx} = 0$$

and to the curve $f(x,y)=0$ at (x,y) is

$$\frac{X-x}{f_x} = \frac{Y-y}{f_y}.$$

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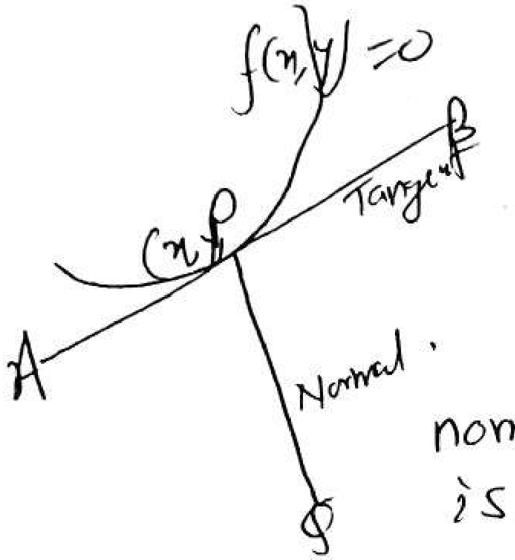
Proof → We know that the equation of the tangent to the curve $y=f(x)$ at a point (x,y) is

$$Y-y=\frac{dy}{dx}(X-x)$$

Here $\frac{dy}{dx}$ is the slope. Let PQ be the normal.



Now slope of the normal

$$=\frac{-1}{\frac{dy}{dx}}$$

Hence the equation to the normal at the point (x,y) is

$$Y-y=\frac{-1}{\frac{dy}{dx}}(X-x)$$

or $(Y-y)\frac{dy}{dx}=-(X-x)$

or $(X-x)+(Y-y)\frac{dy}{dx}=0$

We know that the equation of tangent to the curve $f(x,y)=0$ is

$$(X-x)f_x+(Y-y)f_y=0$$

or $(Y-y)f_y=(-)(X-x)f_x$

or $(Y-y)=(-)(X-x)\frac{f_x}{f_y}$

 So we see that the slope is $\frac{-f_x}{f_y}$.

So the slope to the normal is $\frac{f_y}{f_x}$.

Hence the equation of the normal is

$$Y - y = \frac{f_y}{f_x}(X - x)$$

i.e. $$\frac{X - x}{f_x} = \frac{Y - y}{f_y}.$$

(*) Prove that the

(i) Cartesian Sub-tangent $= \frac{y}{y_1}$

(ii) Cartesian Sub-normal $= y y_1$

(iii) the length of the tangent $= \frac{y}{y_1}\sqrt{1 + y_1^2}$

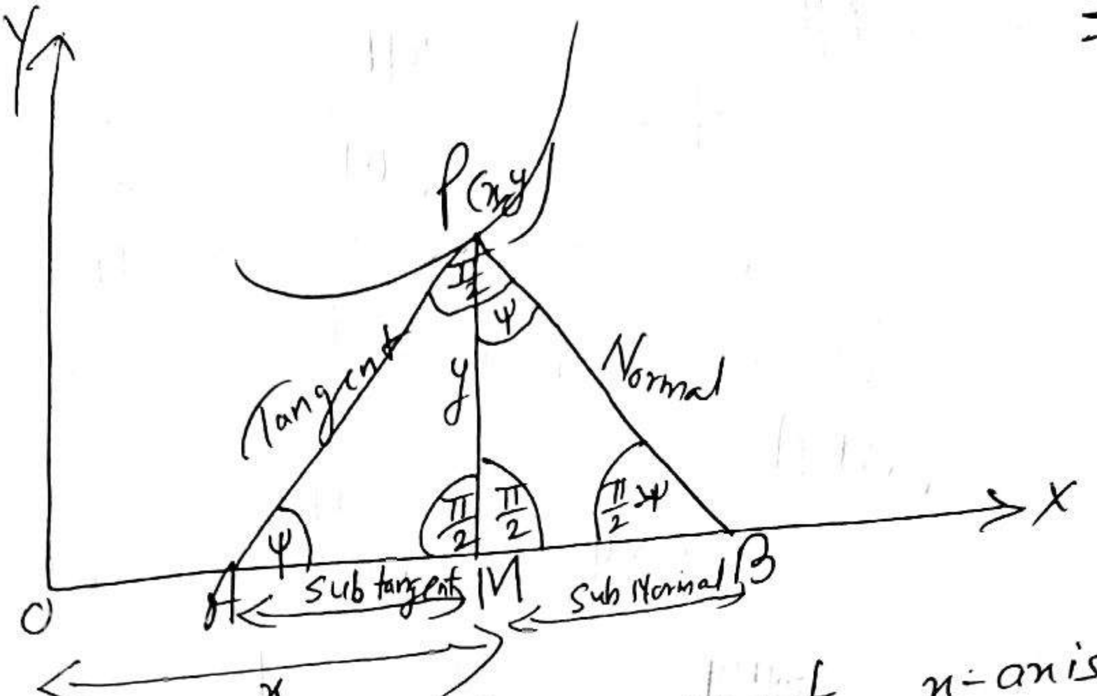
(iv) the length of the normal $= y\sqrt{1 + y_1^2}$

where $y_1 = \frac{dy}{dx}$

Proof: Let $P(x, y)$ be any point on the curve $y = f(x)$

Draw a perpendicular PM to x-axis.

then $OM = x$ & $PM = y$.



Tangent of the Curve meets x-axis on A.
Normal at the Curve meets x-axis on B.
AM is called Subtangent at P.
MB is called sub normal at P.
Here PA is called length of tangent and PB is called the length of the normal.

$$\angle PAB = \psi, \quad \angle BPA = \frac{\pi}{2}$$

$$\therefore \angle PBA = \frac{\pi}{2} - \psi$$

$$\angle BPM = \psi$$

where $\tan\psi = \frac{dy}{dx}$

In $\triangle PMA$ $\quad \tan\psi = \frac{PM}{MA} = \frac{y}{MA}$

$$\text{or } MA = \frac{y}{\tan\psi} = \frac{y}{\frac{dy}{dx}} = \frac{y}{y_1}$$

So Cartesian subtangent $= \frac{y}{y_1}$ Proved

In $\triangle PMB$,

$$\tan\psi = \frac{MB}{PM} = \frac{CNB}{y}$$

$$CNB = y\tan\psi = y\cdot\frac{dy}{dx} = yy_1$$

$\therefore$ Cartesian Sub-normal $= yy_1$ Proved

In $\triangle PMA$,

$$\sin\psi = \frac{PM}{P} = \frac{y}{P}$$

$$P = y\cdot\frac{1}{\sin\psi} = y\operatorname{cosec}\psi = y\sqrt{1+\cot^2\psi}$$

$$= y\sqrt{1+\frac{1}{\tan^2\psi}} = y\sqrt{1+\frac{1}{y_1^2}}$$

$$= y\sqrt{\frac{1+y_1^2}{y_1^2}} = \frac{y}{y_1}\sqrt{1+y_1^2}$$

so P = Length of the tangent $= \frac{y}{y_1}\sqrt{1+y_1^2}$

Proved

Again in $\triangle PMB$

$$\sec\psi = \frac{PB}{PM} = \frac{PB}{y}$$

$$\therefore PB = y\sec\psi = y\sqrt{1+\tan^2\psi} = y\sqrt{1+y_1^2}$$

So, PB = Length of normal $= y\sqrt{1+y_1^2}$

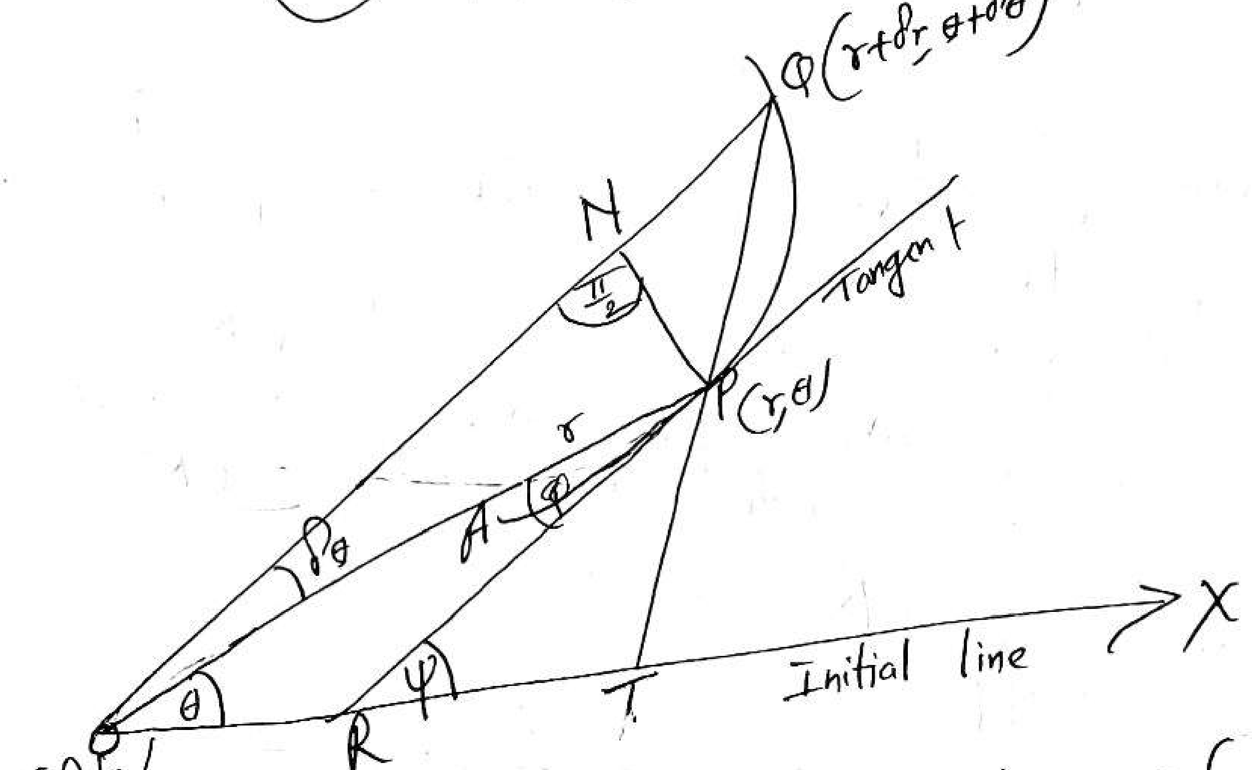
Proved

89 (*) Prove that D.C. 89

(i) $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

(ii) $\frac{ds}{dr} = \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}$

(iii) $\tan\phi = r\frac{d\theta}{dr}$



Let $r = f(\theta)$ be the equation of the curve.
Let O be the pole and OX be the initial line.
Let A be the fixed point on the curve
Length of arc is measured from A.
Let P(r, θ) be a point on the curve and
let arc AP = s
Join O and P. then we have
OP = r, ∠POX = θ

88 Let Q be a point on the curve very close to P and the arc AQ. D.C.88

Arc $AQ = s + \delta s$.

Co-ordinates of Q be $(r+\delta r, \theta + \delta\theta)$

Join O and Q.

So $OQ = r + \delta r$, $\angle QOX = \theta + \delta\theta$.

Now $\angle POQ = \angle QOX - \angle POX$

$= \theta + \delta\theta - \theta$

$= \delta\theta$.

After Joining P and Q we draw a perpendicular PN from P to OQ.

Again we draw a tangent PR at P to the curve. PR makes an angle ψ with the initial line OX. $\angle PRT = \psi$

Let $\angle OPR = \phi$.

In $\triangle ONP$,

$$\cos\delta\theta = \frac{ON}{OP} = \frac{ON}{r}$$

$$\therefore ON = r\cos\delta\theta \quad \text{--- (1)}$$

Again

$$\sin\delta\theta = \frac{PN}{OP} = \frac{PN}{r}$$

$$\text{or } PN = r\sin\delta\theta \quad \text{--- (2)}$$

Now in $\triangle PNQ$

$$PQ^2 = PN^2 + NQ^2$$

$$\text{or } PQ^2 = (r\sin\delta\theta)^2 + (OQ - ON)^2 \quad [\text{using (2)}]$$

$$= r^2\sin^2\delta\theta + (r + \delta r - r\cos\delta\theta)^2$$

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$$PQ^2 = r^2\sin^2\delta\theta + \left[r(1-\cos\delta\theta)+\delta r\right]^2$$

$$= r^2\sin^2\delta\theta + \left[r.\ 2\sin^2\frac{\delta\theta}{2} + \delta r\right]^2$$

or $$\frac{PQ^2}{(\delta\theta)^2} = \frac{r^2\sin^2\delta\theta + \left[2r\sin^2\frac{\delta\theta}{2}+\delta r\right]^2}{(\delta\theta)^2}$$

$$\left(\frac{PQ}{\delta s}\cdot\frac{\delta s}{\delta\theta}\right)^2 = r^2\frac{\sin^2\delta\theta}{(\delta\theta)^2} + \left(\frac{2r\sin^2\frac{\delta\theta}{2}+\delta r}{\delta\theta}\right)^2$$

or $$\left(\frac{PQ}{\delta s}\cdot\frac{\delta s}{\delta\theta}\right)^2 = r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2 + \left[2r\frac{\sin^2\frac{\delta\theta}{2}}{\delta\theta} + \frac{\delta r}{\delta\theta}\right]^2$$

$$= r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2 + \left[2r\cdot\left(\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2\cdot\frac{\delta\theta}{4_2} + \frac{\delta r}{\delta\theta}\right]^2$$

$$= r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2 + \left[\frac{r\,\delta\theta}{2}\left(\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2 + \frac{\delta r}{\delta\theta}\right]^2$$

Let $Q \to P$, we also assume $\frac{\text{Chord } PQ}{\text{Arc } PQ} \to 1$

So that $\delta\theta \to 0$,

$$\therefore \underset{\delta\theta\to 0}{Lt}\left(1\cdot\frac{ds}{d\theta}\right)^2 = r^2\left[\underset{\delta\theta\to 0}{Lt}\frac{\sin\delta\theta}{\delta\theta}\right]^2 + \left[\frac{1}{2}r\underset{\delta\theta\to 0}{Lt}\delta\theta\left(\underset{\delta\theta\to 0}{Lt}\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2 + \underset{\delta\theta\to 0}{Lt}\frac{\delta r}{\delta\theta}\right]^2$$

or $$\left(\frac{ds}{d\theta}\right)^2 = r^2\cdot 1 + \left[\frac{1}{2}r\cdot 0\cdot 1 + \frac{dr}{d\theta}\right]^2$$

or $$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Proved

(ii) Also, $\frac{ds}{dr} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dr}$

or $\frac{ds}{dr} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot \sqrt{\left(\frac{d\theta}{dr}\right)^2}$

$$= \sqrt{r^2\left(\frac{d\theta}{dr}\right)^2 + \left(\frac{dr}{d\theta}\right)^2 \cdot \left(\frac{d\theta}{dr}\right)^2}$$

$$= \sqrt{r^2\left(\frac{d\theta}{dr}\right)^2 + 1}$$

$$\frac{ds}{dr} = \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}$$ Proved

(iii) Let ϕ be the angle between the tangent and the radius vector at any point (r, θ) on the curve $r = f(\theta)$

From equ^n (2) $PN = r\sin\delta\theta$

$$QN = OQ - ON = r + \delta r - r\cos\delta\theta$$
$$= r(1-\cos\delta\theta) + \delta r$$
$$= r \cdot 2\sin^2\frac{\delta\theta}{2} + \delta r$$

In $\triangle PQN$.

$$\tan PQN = \frac{PN}{QN} = \frac{r\sin\delta\theta}{\delta r + 2r\sin^2\frac{\delta\theta}{2}}$$

$$= \frac{r\,\frac{\sin\delta\theta}{\delta\theta}}{\frac{\delta r + 2r\sin^2\frac{\delta\theta}{2}}{\delta\theta}}$$

$$\tan PQN = \frac{\dfrac{r\sin\delta\theta}{\delta\theta}}{\dfrac{\delta r}{\delta\theta} + 2r\cdot\dfrac{\sin^2\frac{\delta\theta}{2}}{\delta\theta}}$$

$$= \frac{\dfrac{r\sin\delta\theta}{\delta\theta}}{\dfrac{\delta r}{\delta\theta} + \cancel{2}r\cdot\left(\dfrac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2\cdot\dfrac{\delta\theta}{\cancel{4}\,2}}$$

$$= \frac{\dfrac{r\sin\delta\theta}{\delta\theta}}{\dfrac{\delta r}{\delta\theta} + \dfrac{1}{2}r\,\delta\theta\left(\dfrac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)^2}$$

Let $Q \to P$ so that $\delta\theta \to 0$

and $\angle PQN \to \angle RPQ = \phi$

$$\therefore \tan\phi = \frac{r\cdot 1}{\dfrac{dr}{d\theta} + \dfrac{1}{2}r\cdot 0\cdot 1}$$

$$= \frac{r}{\dfrac{dr}{d\theta}}$$

or $\tan\phi = r\dfrac{d\theta}{dr}$