(✱) If $y = (x^2-1)^n$

prove that

$$(x^2-1)\,y_{n+2} + 2x \cdot y_{n+1} - n(n+1)\,y_n = 0$$

Ans → Given that

$$y = (x^2-1)^n \quad \ldots\ldots (1)$$

D.c.w.r to x

$$y_1 = n(x^2-1)^{n-1} \cdot 2x = n(x^2-1)^{n-1} \cdot 2x \cdot \frac{(x^2-1)}{(x^2-1)}$$

$$= \frac{2nx\,(x^2-1)^n}{x^2-1}$$

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or $y_1(x^2-1) = 2nx\cdot y$ [from eqn (i)] D.C. 16

D.C. w.r to x

$y_2(x^2-1)+2x\cdot y_1 = 2nx\cdot y_1 + 2ny$

or $y_2(x^2-1)+2xy_1 - 2nxy_1 - 2ny = 0$

or $y_2(x^2-1)+2xy_1(1-n) - 2ny = 0$

D.C. n times w.r.t x by Leibnitz's Theorem

$$y_{n+2}(x^2-1) + \frac{n}{1} y_{n+1}(2x) + \frac{n(n-1)}{\cancel{2}} y_n (\cancel{2})$$

$$+2x\, y_{n+1}(1-n) + \frac{n(1-n)}{1}(2)\, y_n - 2ny_n = 0$$

or $(x^2-1)y_{n+2} + 2nx\, y_{n+1} + n(n-1)y_n$

$+ 2x\, y_{n-1} - 2nx\, y_{n+1} + 2n(1-n)y_n - 2ny_n = 0$

or $(x^2-1)y_{n+2} + 2x\, y_{n+1} + y_n\left[n^2-n+\cancel{2n}-2n^2-\cancel{2n}\right] = 0$

or $(x^2-1)y_{n+2} + 2x\cdot y_{n+1} - n(n+1)y_n = 0$

(4) If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that

(i) $(x^2-1)y_2 + xy_1 - m^2y = 0$

and (ii) $(x^2-1)y_{n+2} + (2n+1)x\cdot y_{n+1} + (n^2-m^2)\cdot y_n = 0$

17 Given that D.C-17

$$y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x \quad \cdots (1)$$

D.C. w.r to x

$$\frac{1}{m} y^{\frac{1}{m}-1} \cdot y_1 - \frac{1}{m} y^{-\frac{1}{m}-1} \cdot y_1 = 2$$

or $\frac{1}{m} y_1 \left[\frac{y^{\frac{1}{m}}}{y} - \frac{y^{-\frac{1}{m}}}{y}\right] = 2$

or $\frac{y_1}{my}\left[y^{\frac{1}{m}} - y^{-\frac{1}{m}}\right] = 2$

or $y_1\left[y^{\frac{1}{m}} - y^{-\frac{1}{m}}\right] = 2my$

squaring

or $y_1^2\left[y^{\frac{1}{m}} - y^{-\frac{1}{m}}\right]^2 = (2my)^2$

or $y_1^2\left[\left(y^{\frac{1}{m}} + y^{-\frac{1}{m}}\right)^2 - 4 \cdot y^{\frac{1}{m}} \cdot y^{-\frac{1}{m}}\right] = 4m^2y^2$

or $y_1^2\left[(2x)^2 - 4\right] = 4m^2y^2$ [using eqn (1)]

or $4\left[x^2y_1^2 - y_1^2\right] = 4m^2y^2$

or $y_1^2(x^2-1) = m^2y^2$

D.C. w.r to x

$$2y_1y_2(x^2-1) + y_1^2 \cdot 2x = m^2 \cdot 2y\,y_1$$

$$y_2(x^2-1) + xy_1 = m^2y$$

or $(x^2-1)y_2 + xy_1 - m^2y = 0$ (i) Proved (2)

Differentiating equation (2) n times w.r. to x

$$(x^2-1)\,y_{n+2} + {}^{n}C_1\,(2x)\,y_{n+1} + {}^{n}C_2\,(2)\,y_n$$
$$+ x\,y_{n+1} + {}^{n}C_1\,(1)\,y_n - m^2 y_n = 0$$

or $(x^2-1)\,y_{n+2} + x.y_{n+1}\left[2\,{}^{n}C_1 + 1\right] + y_n\left[2\,{}^{n}C_2 + {}^{n}C_1 - m^2\right] = 0$

or $(x^2-1)y_{n+2} + x.y_{n+1}(2n+1) + y_n\left[\cancel{2}.\frac{n(n-1)}{\cancel{2}} + n - m^2\right] = 0$

or $(x^2-1)y_{n+2} + x.y_{n+1}(2n+1) + y_n(n^2-m^2) = 0$ (ii) Proved.

(4) If $y = e^{a\sin^{-1}x}$ prove that

(i) $(1-x^2)y_2 - xy_1 - a^2y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

Ans, Given that

$$y = e^{a\sin^{-1}x} \quad \ldots (1)$$

D.C. w.r. to x

$$y_1 = e^{a\sin^{-1}x} \times \frac{a\cdot 1}{\sqrt{1-x^2}} \quad \ldots (2)$$

or $\sqrt{1-x^2}\cdot y_1 = a\,e^{a\sin^{-1}x} = ay$ [from (1)]

Squaring

$$(1-x^2)\cdot y_1^2 = a^2y^2$$

19 D.C. w.r to x D.C-19

$$(1-x^2) 2y_1 y_2 + (-2x) \cdot y_1^2 = a^2 \cdot 2yy_1$$

or $$(1-x^2)y_2 - xy_1 = a^2 y$$

or $$(1-x^2)y_2 - xy_1 - a^2 y = 0 \quad \cdots\cdots (3)$$

Differentiating n times w.r. to x

$$(1-x^2)y_{n+2} + {}^nC_1 (-2x) y_{n+1} + {}^nC_2 (-2) y_n$$
$$- x y_{n+1} - {}^nC_1 (1) y_n - a^2 y_n = 0$$

or $$(1-x^2) y_{n+2} - x \cdot y_{n+1} (2\,{}^nC_1 + 1) - y_n \left[2\,{}^nC_2 - {}^nC_1 - a^2\right] = 0$$

or $$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - y_n \left[\cancel{2} \cdot \frac{n(n-1)}{\cancel{2}} - n - a^2\right] = 0$$

or $$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - y_n (n^2 - a^2) = 0 \quad (ii) \text{ Proved.}$$

(4) If $y = a\cos(\log x) + b\sin(\log x)$ prove that

(i) $x^2 y_2 + x y_1 + y = 0$

(ii) $x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$

Ans → Given that

$$y = a\cos(\log x) + b\sin(\log x)$$

D.C. w.r to x

$$y_1 = -a\sin(\log x) \cdot \frac{1}{x} + b\cos(\log x) \cdot \frac{1}{x}$$

2^o or $xy_1 = -a\sin(\log x) + b\cos(\log x)$ $\underline{DE \cdot 20}$

D.C. w.r.to x

$$xy_2 + y_1 = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$$

or $x^2y_2 + xy_1 = -a\cos(\log x) - b\sin(\log x)$

$$= -[a\cos(\log x) + b\sin(\log x)]$$

$$= -y$$

or $x^2y_2 + xy_1 + y = 0$ ....(1) (i) Proved

D.C. eqn (1) n times w.r.to x

$$x^2 y_{n+2} + {}^nc_1(2x)\cdot y_{n+1} + {}^nc_2(2)y_n + y_{n+1}x$$

$$+ {}^nc_1\, y_n(1) + y_n = 0$$

or $x^2\cdot y_{n+2} + x\cdot y_{n+1}[2\,{}^nc_1 + 1] + y_n[2\,{}^nc_2 + {}^nc_1 + 1] = 0$

or $x^2\cdot y_{n+2} + x\cdot y_{n+1}(2n+1) + y_n\left[\frac{\cancel{2}\cdot n(n-1)}{\cancel{2}} + n + 1\right] = 0$

or $x^2\cdot y_{n+2} + x\cdot y_{n+1}(2n+1) + y_n(n^2 - \cancel{n} + \cancel{n} + 1) = 0$

or $x^2\cdot y_{n+2} + x\cdot y_{n+1}(2n+1) + (n^2+1)y_n = 0$ (ii) proved

21 (4) If $y = \sin(m\sin^{-1}x)$ prove that D-C-21

(i) $(1-x^2)y_2 - xy_1 + m^2y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Ans Given that

$$y = \sin(m\sin^{-1}x)$$

or $\sin^{-1}y = m\sin^{-1}x$

D.C. w.r. to x

$$\frac{1}{\sqrt{1-y^2}}\, y_1 = m\frac{1}{\sqrt{1-x^2}}$$

or $y_1\sqrt{1-x^2} = m\sqrt{1-y^2}$

Squaring

$$y_1^2(1-x^2) = m^2(1-y^2)$$

D.C. w.r to x

$$2y_1y_2(1-x^2) + y_1^2(-2x) = m^2(-2y)y_1$$

or $2y_1\left[(1-x^2)y_2 - xy_1\right] = (-)\cdot m^2\cdot 2yy_1$

or $y_2(1-x^2) - xy_1 + m^2y = 0$ .... (i) Proved

D.C. n times w.r. to x by Leibnitz's theorem, we get

$$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) -$$
$$\left[y_{n+1}\cdot x + {}^nC_1\, y_n\right] + m^2y_n = 0$$

22 D.C.22

or $y_{n+2}(1-x^2) + n \cdot y_{n+1}(-2x) + \frac{n(n-1)}{\cancel{2}} y_n (-\cancel{2})$

$- y_{n+1} \cdot x - n \cdot y_n + m^2 y_n = 0$

or $(1-x^2) y_{n+2} - x \cdot y_{n+1}(2n+1) + y_n [m^2 - \cancel{n} - n^2 + \cancel{n}] = 0$

or $(1-x^2) y_{n+2} - x \cdot y_{n+1}(2n+1) + y_n (m^2 - n^2) = 0$ .....(ii) proved

(7) If $y = \sin^{-1}x$ prove that

(i) $(1-x^2) y_2 - x y_1 = 0$

and (ii) $(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0$

Ans → Given that

$y = \sin^{-1}x$

D.C. w.r. to x

$y_1 = \frac{1}{\sqrt{1-x^2}}$

or $\sqrt{1-x^2} \cdot y_1 = 1$

D.C. w.r. to x

$\sqrt{1-x^2} \cdot y_2 + \frac{1}{\cancel{2}\sqrt{1-x^2}} \cdot (-\cancel{2}x) y_1 = 0$

or $(1-x^2) y_2 - x y_1 = 0$ .....(1) proved

D.C. n times w.r. to x by Leibnitz's theorem

$y_{n+2}(1-x^2) + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n (-2)$

$- y_{n+1} x - {}^nC_1 y_n \cdot (1) = 0$

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$$\text{or } (1-x^2)\,y_{n+2} + x\cdot y_{n+1}(-2n-1) - \frac{n(n-1)}{\cancel{2}}\,y_n\,\cancel{(-2)} - n\,y_n = 0 \qquad \text{D.C. 23}$$

$$\text{or } (1-x^2)\,y_{n+2} - x\,y_{n+1}(1+2n) + y_n(n^2-n) - n\,y_n = 0$$

$$\text{or } (1-x^2)\,y_{n+2} - (1+2n)\cdot x\,y_{n+1} + n^2\cdot y_n = 0 \;\cdots\text{(ii)}$$

proved

(7) If $y = \tan^{-1}x$ prove that

$$(1+x^2)\,y_{n+1} + 2nx\,y_n + n(n-1)\,y_{n-1} = 0$$

Ans:

Given that

$$y = \tan^{-1}x$$

D.C. w.r. to x

$$y_1 = \frac{1}{1+x^2}$$

$$\text{or } (1+x^2)\cdot y_1 = 1$$

D.C. n times w.r. to x by Leibnitz's theorem we get

$$y_{n+1}(1+x^2) + {}^nc_1\,y_n\cdot(2x) + {}^nc_2\,y_{n-1}\cdot(2) = 0$$

$$\text{or } (1+x^2)\cdot y_{n+1} + 2nx\,y_n + \frac{n(n-1)}{\cancel{2}}\cdot y_{n-1}\cdot\cancel{2} = 0$$

$$\text{or } (1+x^2)\cdot y_{n+1} + 2nx\,y_n + n(n-1)\,y_{n-1} = 0$$