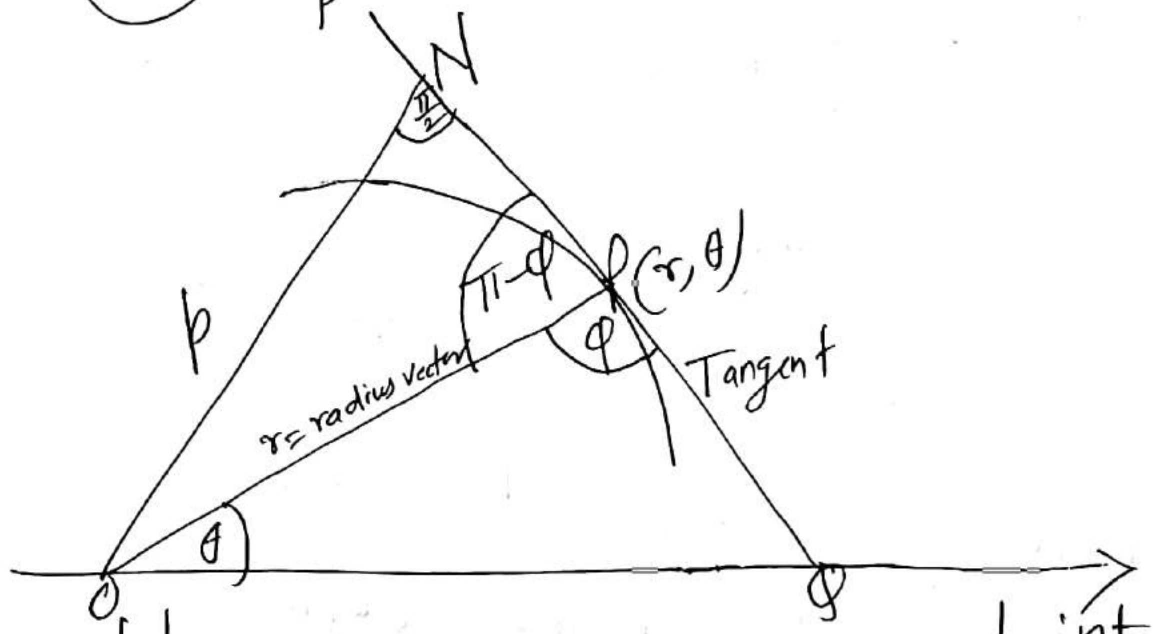
92 (4) Prove that (i) $p = r\sin\phi$ D.C.92

(ii) $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

(iii) $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$



Let $P(r,\theta)$ be a given point on the curve. Join O and P.
$OP = r$

Draw a perpendicular ON from O on the tangent.

Let $ON = p$, Let $\angle OP\mathscr{G} = \phi$

$\therefore \angle OPN = \pi - \phi$.

In $\triangle OPN$, $\sin(\pi-\phi) = \frac{ON}{OP} = \frac{p}{r}$

or $\sin\phi = \frac{p}{r}$

or $p = r\sin\phi$ ......(1) Proved

or $\frac{1}{p} = \frac{1}{r\sin\phi}$

squaring

$$\frac{1}{p^2} = \frac{1}{r^2 \sin^2\phi}$$

or $$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}(1+\cot^2\phi)$$

or $$\frac{1}{p^2} = \frac{1}{r^2}\left(1+\frac{1}{\tan^2\phi}\right)$$

we know that $\tan\phi = r\frac{d\theta}{dr}$

or $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2}\cdot\frac{1}{\tan^2\phi}$$

$$= \frac{1}{r^2} + \frac{1}{r^2}\cdot\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$ (ii) proved ---- (1)

Let $u = \frac{1}{r}$

D.C. w.r. to θ

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

or $$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

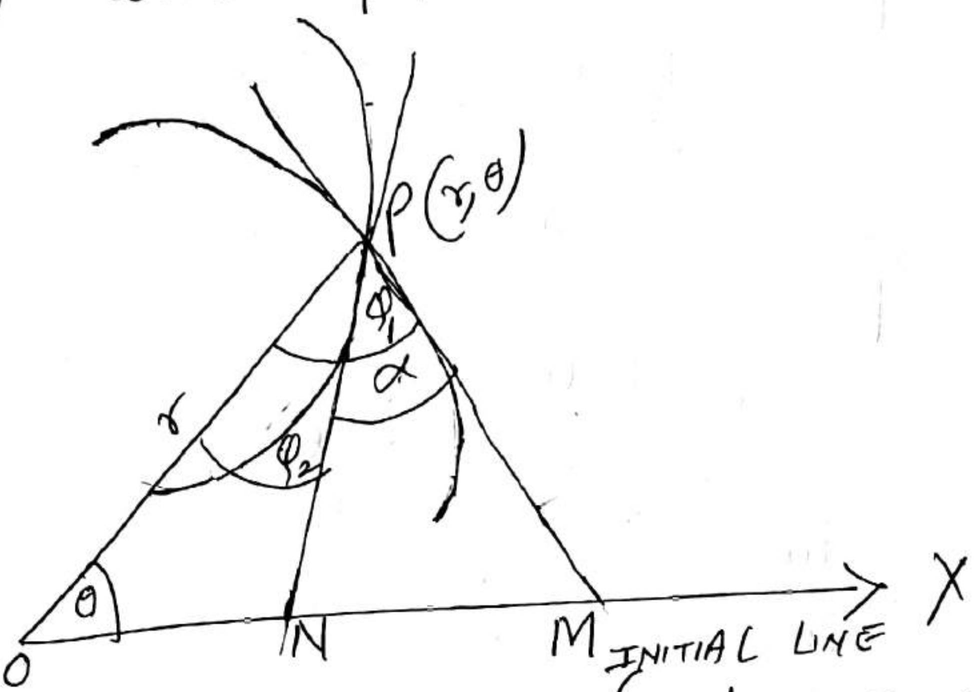
Putting this value in equation (1) we get

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$ (iii). proved

PEDAL EQUATION →

Relation between the perpendicular on the tangent from a given point and the radius vector of the point of contact from the given point is called pedal equation of the given curve.

(*) Find the angle of intersection of two curves whose polar equations are given.



Let the polar equations of two curves be given as, $r = f(\theta)$ and $r = F(\theta)$.

Let the two curves intersect each other at point $P(r, \theta)$.

Join O and P then $OP = r$, $\angle POX = \theta$.

$\because r = f(\theta)$

D.C. w.r. to θ

$$\frac{dr}{d\theta} = f'(\theta)$$

15 $r = F(\theta)$ D-C-95

D.C. w.r.to θ

$$\frac{dr}{d\theta} = F'(\theta)$$

Draw tangents PM and PN at P to the Curves.

$$\angle OPM = \phi_1, \quad \angle OPN = \phi_2$$

$$\tan\phi_1 = r\frac{d\theta}{dr} = r \cdot \frac{1}{\frac{dr}{d\theta}} = \frac{f(\theta)}{f'(\theta)}$$

$$\tan\phi_2 = r\frac{d\theta}{dr} = r \cdot \frac{1}{\frac{dr}{d\theta}} = \frac{F(\theta)}{F'(\theta)}$$

we know that the angle of intersection of two curves is the angle between their tangents at the Common point of intersection.

If α be the required angle then

$$\alpha = \phi_1 - \phi_2$$

$$\therefore \tan\alpha = \tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \cdot \tan\phi_2}$$

$$= \frac{\frac{f(\theta)}{f'(\theta)} - \frac{F(\theta)}{F'(\theta)}}{1 + \frac{f(\theta)}{f'(\theta)} \cdot \frac{F(\theta)}{F'(\theta)}}$$

$$\tan\alpha = \frac{\frac{f(\theta)F'(\theta) - F(\theta)f'(\theta)}{\cancel{f'(\theta)F'(\theta)}}}{\frac{f'(\theta)F'(\theta) + f(\theta)F(\theta)}{\cancel{f'(\theta)F'(\theta)}}} = \frac{f(\theta)F'(\theta) - F(\theta)f'(\theta)}{f'(\theta)F'(\theta) + f(\theta)F(\theta)}$$

$$\text{or } \alpha = \tan^{-1}\left\{\frac{f(\theta)F'(\theta) - F(\theta)f'(\theta)}{f'(\theta)F'(\theta) + f(\theta)F(\theta)}\right\}$$

This is the required angle.

Important formulas

(1) Equation of tangent to the curve $y = f(x)$ at a point (x,y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

(2) Equation of normal to the curve $y = f(x)$ at a point (x,y) is

$$(X - x) + (Y - y)\frac{dy}{dx} = 0$$

(3) Equation of tangent at a point (x,y) to the curve $f(x,y) = 0$ is

$$(X - x) f_x + (Y - y) f_y = 0$$

or $$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z} = 0$$

where $z = 1$ and $Z = 1$

(4) Equation of normal to the curve $f(x,y) = 0$ is

$$\frac{X - x}{f_x} = \frac{Y - y}{f_y}$$

(5) Subtangent $= \frac{y}{y_1}$, subnormal $= y y_1$

(6) Length of tangent $= \frac{y}{y_1}\sqrt{1 + y_1^2}$

(7) Length of normal $= y\sqrt{1 + y_1^2}$

(8) $ds^2 = dx^2 + dy^2$

(9) If two conics cut each other orthogonally then $f_x F_x + f_y F_y = 0$

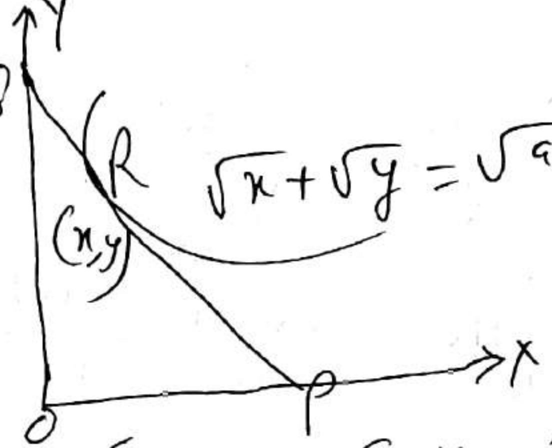
(10) $\frac{dx}{dy} = \pm 1$ [when the normal to the curve makes equal intercepts]

9. (*) Show that the sum of the intercepts of the tangent to $\sqrt{x}+\sqrt{y}=\sqrt{a}$ upon the coordinate axis is Constant. D.C. 97

(or)

If the tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at any point on it cuts the axes OX and OY at P and Q respectively prove that $OP+OQ=$ Constant.

Ans,



The equation of given curve is

$$\sqrt{x}+\sqrt{y}=\sqrt{a} \quad \text{..(1)}$$ where curve is $y=f(x)$

D.C. w.r to x

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\text{or } \frac{1}{\cancel{2}\sqrt{y}}\frac{dy}{dx}=-\frac{1}{\cancel{2}\sqrt{x}}$$

$$\text{or } \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

We know that the equation of the tangent to the curve $y=f(x)$ at point (x,y) is

$$Y-y=\frac{dy}{dx}(X-x)$$

$$\text{So, } Y-y=-\frac{\sqrt{y}}{\sqrt{x}}(X-x)$$

or $\frac{X-x}{\sqrt{x}} = (-)\frac{Y-y}{\sqrt{y}}$

or $\frac{X}{\sqrt{x}} - \sqrt{x} = (-)\frac{Y}{\sqrt{y}} + \sqrt{y}$

or $\frac{X}{\sqrt{x}} + \frac{Y}{\sqrt{y}} = \sqrt{x}+\sqrt{y} = \sqrt{a}$ [from equ^n (1)]

or $\frac{X}{\sqrt{ax}} + \frac{Y}{\sqrt{ay}} = 1$

$\therefore$ $OP = \sqrt{ax}$, $OQ = \sqrt{ay}$

Sum of the intercepts $= OP + OQ$

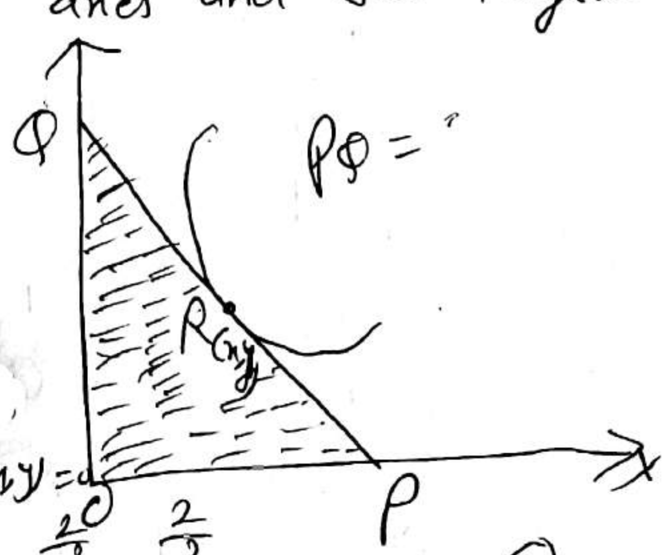
$= \sqrt{ax} + \sqrt{ay}$

$= \sqrt{a}\,(\sqrt{x}+\sqrt{y})$

$= \sqrt{a}\cdot\sqrt{a}$

$= a$ which is constant.

(*) Show that the portion of the tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ which is intercepted between the axes is of constant length and find the area of the portion included between the axes and the tangents



Ans: The given equation of the curve is

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

or $x^{\frac{2}{3}} + y^{\frac{2}{3}} - a^{\frac{2}{3}} = 0$

Curve is of the form $f(x,y)=0$

Equation of curve $f(x,y) = x^{\frac{2}{3}} + y^{\frac{2}{3}} - a^{\frac{2}{3}} = 0$ .....(1)

D.C. equation (1) w.r.to x

$$f_x = \frac{2}{3} x^{-\frac{1}{3}}$$

Again d.c. (1) w.r.to y

$$f_y = \frac{2}{3} y^{-\frac{1}{3}}$$

we know that the equation of tangent to the curve $f(x, y) = 0$ is

$$(X - x) f_x + (Y - y) f_y = 0$$

or $(X - x) \cdot \frac{2}{3} x^{-\frac{1}{3}} + (Y - y) \cdot \frac{2}{3} y^{-\frac{1}{3}} = 0$

or $X \cdot x^{-\frac{1}{3}} - x^{\frac{2}{3}} + Y \cdot y^{-\frac{1}{3}} - y^{\frac{2}{3}} = 0$

or $X \cdot x^{-\frac{1}{3}} + Y \cdot y^{-\frac{1}{3}} = x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

or $\frac{X}{x^{\frac{1}{3}}} + \frac{Y}{y^{\frac{1}{3}}} = a^{\frac{2}{3}}$

or $\frac{X}{x^{\frac{1}{3}} \cdot a^{\frac{2}{3}}} + \frac{Y}{y^{\frac{1}{3}} \cdot a^{\frac{2}{3}}} = 1$

or $OP = x^{\frac{1}{3}} \cdot a^{\frac{2}{3}}, \quad OQ = y^{\frac{1}{3}} \cdot a^{\frac{2}{3}}$

In $\triangle OPQ$ $\quad PQ^2 = OP^2 + OQ^2$

or $PQ = \sqrt{OP^2 + OQ^2} = \sqrt{\left(x^{\frac{1}{3}} \cdot a^{\frac{2}{3}}\right)^2 + \left(y^{\frac{1}{3}} \cdot a^{\frac{2}{3}}\right)^2}$

$= a^{\frac{2}{3}} \sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}} = a^{\frac{2}{3}} \cdot \sqrt{a^{\frac{2}{3}}}$

or $PQ = a^{\frac{2}{3}} \cdot a^{\frac{1}{3}} = a = \text{constant}$

Again,
Area of the portion included between the axes and the tangent

$$= \text{Area of the } \triangle OPQ$$

$$= \frac{1}{2} OP.OQ$$

$$= \frac{1}{2}\left(a^{\frac{2}{3}}.x^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}}.y^{\frac{1}{3}}\right)$$

$$= \frac{1}{2} a^{\frac{4}{3}}.x^{\frac{1}{3}}.y^{\frac{1}{3}}$$

(4) If $p = x\cos\alpha + y\sin\alpha$ touches the Curve $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$, prove that

$$p^{\frac{m}{m-1}} = (a\cos\alpha)^{\frac{m}{m-1}} + (b\sin\alpha)^{\frac{m}{m-1}}.$$

Ans, The given equation of the Curve is

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$$

D.C. w.r.to x

$$\frac{mx^{m-1}}{a^m} + \frac{my^{m-1}}{b^m}\frac{dy}{dx} = 0$$

or $\frac{dy}{dx}\cdot\frac{\cancel{m}y^{m-1}}{b^m} = (-)\frac{\cancel{m}x^{m-1}}{a^m}$

or $\frac{dy}{dx} = (-)\frac{b^m}{a^m}\frac{x^{m-1}}{y^{m-1}}$

So the equation of the curve is

$$\frac{dy}{dx} = (-)\frac{b^m}{a^m}\frac{x^{m-1}}{y^{m-1}} \quad \cdots (1)$$

At the point (x_1, y_1) of the curve

$$\frac{dy}{dx} = (-)\frac{b^m}{a^m}\frac{x_1^{m-1}}{y_1^{m-1}} \quad \cdots (2)$$

We know that the tangent of the curve $y = f(x)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

So at point (x_1, y_1) the equation of the tangent is

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

or $$y - y_1 = (-)\frac{b^m}{a^m}\frac{x_1^{m-1}}{y_1^{m-1}}(x - x_1) \quad [\text{from } (2)]$$

or $$\frac{(y - y_1)y_1^{m-1}}{b^m} = \frac{(-)x_1^{m-1}(x - x_1)}{a^m}$$

or $$\frac{y y_1^{m-1} - y_1^m}{b^m} = (-)\frac{x \cdot x_1^{m-1}}{a^m} + \frac{x_1^m}{a^m}$$

or $$\frac{y \cdot y_1^{m-1}}{b^m} - \frac{y_1^m}{b^m} = \frac{(-)x \cdot x_1^{m-1}}{a^m} + \frac{x_1^m}{a^m}$$

or $$\frac{y \cdot y_1^{m-1}}{b^m} + \frac{x \cdot x_1^{m-1}}{a^m} = \frac{x_1^m}{a^m} + \frac{y_1^m}{b^m} \quad \cdots (3)$$

102 Since the point (x_1, y_1) lies on the D.C.102 curve, so the equation of the curve is

$$\frac{x_1^m}{a^m} + \frac{y_1^m}{b^m} = 1 \quad \cdots\cdots (4)$$

from equation (3)

$$\frac{x \cdot x_1^{m-1}}{a^m} + \frac{y \cdot y_1^{m-1}}{b^m} = 1 \quad \cdots (5)$$

Given Equation of the tangent

$$x\cos\alpha + y\sin\alpha = p$$

$$\text{or} \quad \frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1 \quad \cdots\cdots (6)$$

Since Equation (5) & (6) represent the same tangent, so comparing them we get

$$\frac{x_1^{m-1}}{a^m} = \frac{\cos\alpha}{p}$$

$$\& \quad \frac{y_1^{m-1}}{b^m} = \frac{\sin\alpha}{p}$$

$$\therefore \frac{x_1^{m-1}}{a^m} = \frac{\cos\alpha}{p}$$

$$\text{or } x_1^{m-1} = \frac{a^m \cos\alpha}{p}$$

$$\text{or } x_1 = \frac{a^{\frac{m}{m-1}} (\cos\alpha)^{\frac{1}{m-1}}}{p^{\frac{1}{m-1}}}$$

$$\therefore \frac{y_1^{m-1}}{b^m} = \frac{\sin\alpha}{p}$$

$$\therefore y_1^{m-1} = \frac{b^m \sin\alpha}{p}$$

$$\text{or } y_1 = \frac{b^{\frac{m}{m-1}} (\sin\alpha)^{\frac{1}{m-1}}}{p^{\frac{1}{m-1}}}$$

103 Now from equation (4) DC-103

$$\frac{x_1^m}{a^m} + \frac{y_1^m}{b^m} = 1$$

or $$\frac{a^{\frac{m^2}{m-1}} (\cos\alpha)^{\frac{m}{m-1}}}{p^{\frac{m}{m-1}} \cdot a^m} + \frac{b^{\frac{m^2}{m-1}} (\sin\alpha)^{\frac{m}{m-1}}}{p^{\frac{m}{m-1}} \, b^m} = 1$$

or $$a^{\frac{m^2}{m-1} - m} \cdot (\cos\alpha)^{\frac{m}{m-1}} + b^{\frac{m^2}{m-1} - m} \cdot (\sin\alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$$

or $$a^{\frac{\cancel{m^2} - \cancel{m^2} + m}{m-1}} (\cos\alpha)^{\frac{m}{m-1}} + b^{\frac{\cancel{m^2} - \cancel{m^2} + m}{m-1}} \cdot (\sin\alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$$

or $$a^{\frac{m}{m-1}} \cdot (\cos\alpha)^{\frac{m}{m-1}} + b^{\frac{m}{m-1}} \cdot (\sin\alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$$

or $$(a\cos\alpha)^{\frac{m}{m-1}} + (b\sin\alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$$