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$$x\frac{d^2y}{dx^2} - (2x+1)\frac{dy}{dx} + (x+1)y = (x^2+x-1)e^{2x}$$

Ans) Given that

$$x\frac{d^2y}{dx^2} - (2x+1)\frac{dy}{dx} + (x+1)y = (x^2+x-1)e^{2x}$$

or $$\frac{d^2y}{dx^2} - \left(2+\frac{1}{x}\right)\frac{dy}{dx} + \left(\frac{x+1}{x}\right)y = \left(\frac{x^2+x-1}{x}\right)e^{2x} \quad (1)$$

Now Comparing it with standard form we get

$$P = (-)\frac{2x+1}{x}, \quad Q = \frac{x+1}{x}, \quad R = \frac{x^2+x-1}{x}\cdot e^{2x}$$

Here $1+P+Q$

$$= 1 - \frac{2x+1}{x} + \frac{x+1}{x} = \frac{x-2x-1+x+1}{x}$$

$$= 0$$

$\therefore y = e^x$ is a part of C.F. of the solution of equation (1)

Putting $y = v e^x$

$$\frac{dy}{dx} = \frac{dv}{dx}e^x + v e^x$$

& $$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}e^x + 2\frac{dv}{dx}e^x + v e^x$$

Now from equation (1),

$$\frac{d^2v}{dx^2}e^x + 2\frac{dv}{dx}e^x + v\cdot e^x - \left(\frac{2x+1}{x}\right)\left(\frac{dv}{dx}+v\right)e^x + \left(\frac{x+1}{x}\right)v\cdot e^x = \left(\frac{x^2+x-1}{x}\right)e^{2x}$$

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or, $$\frac{d^2v}{dx^2}+\frac{dv}{dx}\left[2-\frac{2x+1}{x}\right]+v-\frac{2x+1}{x}v+\frac{x+1}{x}v=\frac{x^2+x-1}{x}e^x$$

or $$\frac{d^2v}{dx^2}+\left[\frac{2x-2x-1}{x}\right]\frac{dv}{dx}+v\left[\frac{\cancel{x}-2x-1+x+1}{x}\right]=\frac{x^2+x-1}{x}e^x$$

or, $$\frac{d^2v}{dx^2}-\frac{1}{x}\frac{dv}{dx}+0=\frac{x^2+x-1}{x}e^x$$

Now $p=\frac{dv}{dx} \Rightarrow \frac{dp}{dx}=\frac{d^2v}{dx^2}$

$$\frac{dp}{dx}-\frac{1}{x}p=\frac{x^2+x-1}{x}e^x$$

which is linear

$$I.F.=e^{(-)\int\frac{1}{x}dx}=e^{-\log x}=e^{\log\frac{1}{x}}=\frac{1}{x}$$

$$\therefore p\cdot\frac{1}{x}=\int\frac{1}{x}\cdot\frac{x^2+x-1}{x}e^x\,dx+k$$

$$=\int\left(\frac{x^2+x-1}{x^2}\right)e^x dx+k=\int\left[\left(1+\frac{1}{x}\right)-\frac{1}{x^2}\right]e^x dx+k$$

we know that

$$\int e^x[f(x)+f'(x)]dx=e^x f(x)$$

Here $f(x)=1+\frac{1}{x}$

So, $p\cdot\frac{1}{x}=e^x\left(1+\frac{1}{x}\right)+k$

or $p=\frac{dv}{dx}=e^x(1+x)+kx$

or $dv=e^x(1+x)dx+kx\,dx$

Integrating we get

$$\int dv = \int e^x (1+x)\,dx + k\int x\,dx$$

or $$v = \int e^x dx + \int e^x . x\,dx + k\frac{x^2}{2} + k_1$$

$$= \cancel{e^x} + xe^x - \cancel{e^x} + k\frac{x^2}{2} + k_1$$

or $$v = xe^x + k\cdot\frac{x^2}{2} + k_1$$

Hence the complete solution of the given differential equation is

$$y = ve^x$$

or $$y = \left(xe^x + k\frac{x^2}{2} + k_1\right)e^x$$

or $$y = xe^{2x} + \frac{k}{2}\cdot x^2 e^x + k_1 e^x$$

(4) Solve,

$$x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$$

Ans → Given that

$$x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$$

or $$\frac{d^2y}{dx^2} - 2\left(\frac{1}{x}+1\right)\frac{dy}{dx} + 2\left(\frac{1}{x}+\frac{1}{x^2}\right)y = x \quad \ldots (1)$$

Now comparing it with standard form of equation

we have $P = -2\left(\frac{1}{x}+1\right)$, $Q = 2\left(\frac{1}{x}+\frac{1}{x^2}\right)$

$$P + Qx = -2\left(\frac{1}{x}+1\right) + 2x\left(\frac{1}{x^2}+\frac{1}{x}\right)$$
$$= -\cancel{\frac{2}{x}} - \cancel{2} + \cancel{\frac{2}{x}} + \cancel{2} = 0$$

So $y = x$ is a part. of C.F.

Putting $y = vx$

So, $\frac{dy}{dx} = \frac{dv}{dx}x + v$

& $\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}x + 2\frac{dv}{dx}$

Now from equation (1) we get.

$$\frac{d^2v}{dx^2}x + 2\frac{dv}{dx} - 2\left(\frac{1+x}{x}\right)\left(x\cdot\frac{dv}{dx} + v\right) + 2\left(\frac{1}{x^2}+\frac{1}{x}\right)vx = x$$

or $\frac{d^2v}{dx^2} + \frac{dv}{dx}\left\{\frac{2}{x} - \frac{2x(1+x)}{x^2}\right\} - 2(1+x)\cdot\frac{v}{x^2} + 2\left(\frac{1}{x^2}+\frac{1}{x}\right)v = 1$

or $\frac{d^2v}{dx^2} + \frac{dv}{dx}\cdot\frac{2}{x}\left\{\cancel{1}-\cancel{1}-x\right\} + \cancel{2\left(\frac{1+x}{x^2}\right)\cdot v} - \cancel{2\left(\frac{1+x}{x^2}\right)v} = 1$

or $\frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1$

or $\frac{dp}{dx} - 2p = 1$ $\left[\text{where } p = \frac{dv}{dx} \text{ \& } \frac{dp}{dx} = \frac{d^2v}{dx^2}\right]$

which is linear in p.

$$I.F = e^{\int(-)2dx} = e^{-2x}$$

$$\therefore p \cdot e^{-2x} = \int 1 \cdot e^{-2x}\,dx + k$$

$$= -\frac{1}{2}e^{-2x} + k$$

or $$\frac{dv}{dx} = -\frac{1}{2} + k e^{2x}$$

or $$dv = -\frac{1}{2}dx + k e^{2x}dx$$

Integrating

$$\int dv = -\frac{1}{2}\int dx + k\int e^{2x}dx$$

or $$v = -\frac{1}{2}x + k\frac{e^{2x}}{2} + k_1$$

The Complete Solution of equation (1) is

$$y = vx$$

or $$y = -\frac{1}{2}x^2 + \frac{1}{2}kxe^{2x} + k_1 \cdot x$$