(4) If $u = \tan^{-1}\frac{y}{x}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Ans, Given that

$$u = \tan^{-1}\frac{y}{x}$$

or $\tan u = \frac{y}{x}$ ... (1)

D.C. partially w.r to x

$$\sec^2 u \frac{\partial u}{\partial x} = (-)\frac{1}{x^2}\cdot y$$

or $(1+\tan^2 u)\frac{\partial u}{\partial x} = \frac{-y}{x^2}$

or $\frac{\partial u}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot (-)\frac{y}{x^2} = \frac{x^2}{x^2+y^2}\cdot(-)\frac{y}{x^2} = \frac{-y}{x^2+y^2}$

Scanned with CamScanner

So $\frac{\partial u}{\partial x} = \frac{-y}{x^2+y^2}$

$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}$ ... (2)

Again d.c. (1) partially w.r to y

$\sec^2 u \frac{\partial u}{\partial y} = \frac{1}{x}$

or $(1+\tan^2 u)\frac{\partial u}{\partial y} = \frac{1}{x}$

or $\left(1+\frac{y^2}{x^2}\right)\frac{\partial u}{\partial y} = \frac{1}{x}$

or $\frac{\partial u}{\partial y} = \frac{1}{x} \cdot \frac{1}{\frac{x^2+y^2}{x^2}} = \frac{x}{x^2+y^2}$

$\frac{\partial^2 u}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}$ ... (3)

Adding eqn (2) & (3) we get

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$

(4) If ∇^2 stands for the operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, prove that $\nabla^2 \frac{1}{r} = 0$, where $\frac{1}{r^2} = x^2+y^2+z^2$.

Ans Given that

$\frac{1}{r^2} = x^2+y^2+z^2$ ... (1)

D.C. partially w.r. to x

Scanned with CamScanner

$$-\frac{2}{r^3}\cdot\frac{\partial r}{\partial x} = 2x$$

$$\text{or } \frac{\partial r}{\partial x}\cdot\frac{1}{r^3} = (-)x \quad \cdots (2)$$

Again differentiating partially w.r.to x

$$(-)\frac{3r^2}{r^6}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{1}{r^3}\cdot\frac{\partial^2 r}{\partial x^2} = -1$$

$$\text{or } \frac{-3}{r^4}(-xr^3)^2 + \frac{1}{r^3}\frac{\partial^2 r}{\partial x^2} = -1$$

$$\text{or } -3x^2r^2 + \frac{1}{r^3}\frac{\partial^2 r}{\partial x^2} = -1 \quad \cdots (3)$$

Similarly differentiating eqn (1) partially w.r.to y and z respectively we get

$$-3y^2r^2 + \frac{1}{r^3}\frac{\partial^2 r}{\partial y^2} = -1 \quad \cdots (4)$$

$$-3z^2r^2 + \frac{1}{r^3}\frac{\partial^2 r}{\partial z^2} = -1 \quad \cdots (5)$$

Adding equation (3), (4) & (5) we get

$$-3r^2(x^2+y^2+z^2) + \frac{1}{r^3}\left(\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2}\right) = -3$$

$$\text{or } -3r^2\cdot\frac{1}{r^2} + \frac{1}{r^3}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)r = -3$$

$$\text{or } \frac{1}{r^3}\cdot\nabla^2 r = 0$$

$$\text{or } \nabla^2 r = 0$$

Proved

Scanned with CamScanner

50 (4) If $u = \log(x^3+y^3+z^3-3xyz)$ prove that D.C.50

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-3}{(x+y+z)^2}$$

Ans, Given that

$$u = \log(x^3+y^3+z^3-3xyz)$$

$$\text{or } u = \log\left[(x+y+z)(x^2+y^2+z^2-xy-yz-zx)\right]$$

$$u = \log(x+y+z) + \log(x^2+y^2+z^2-xy-yz-zx)$$

D.C. partially w.r. to x

$$\frac{\partial u}{\partial x} = \frac{1}{x+y+z} + \frac{2x-y-z}{x^2+y^2+z^2-xy-yz-zx}$$

Again D.C. partially w.r. to x

$$\frac{\partial^2 u}{\partial x^2} = \frac{-1}{(x+y+z)^2} + \frac{2(x^2+y^2+z^2-xy-yz-zx)-(2x-y-z)(2x-y-z)}{(x^2+y^2+z^2-xy-yz-zx)^2}$$

$$= \frac{-1}{(x+y+z)^2} + \frac{2x^2+2y^2+2z^2-2xy-2yz-2zx-(4x^2+y^2+z^2-4xy+2yz-4zx)}{(x^2+y^2+z^2-xy-yz-zx)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-1}{(x+y+z)^2} + \frac{y^2+z^2+2xy+2xz-2x^2-4yz}{(x^2+y^2+z^2-xy-yz-zx)^2}$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = \frac{-1}{(x+y+z)^2} + \frac{z^2+x^2+2yz+2yx-2y^2-4xz}{(x^2+y^2+z^2-xy-yz-zx)^2}$$

and

$$\frac{\partial^2 u}{\partial z^2} = \frac{-1}{(x+y+z)^2} + \frac{x^2+y^2+2zx+2zy-2z^2-4yx}{(x^2+y^2+z^2-xy-yz-zx)^2}$$

Scanned with CamScanner

5] Adding we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 z}{\partial z^2}=\frac{-3}{(x+y+z)^2}$$

$$+\frac{y^2+z^2+2xy+2xz-2x^2-4yz+z^2+x^2+2yz+2yx-2y^2-4zx+x^2+y^2+2zx+2yz-2z^2-4xy}{(x^2+y^2+z^2-xy-zx-zx)^2}$$

$$=\frac{-3}{(x+y+z)^2}+0=\frac{-3}{(x+y+z)^2}$$

(*) If $u=\log(x^3+y^3+z^3-3xyz)$ prove that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\frac{-9}{(x+y+z)^2}$$

Ans, Given that

$$u=\log(x^3+y^3+z^3-3xyz) \quad \cdots (1)$$

D.C. (1) partially w.r. to x

$$\frac{\partial u}{\partial x}=\frac{1}{x^3+y^3+z^3-3xyz}\cdot(3x^2-3yz)$$

$$\frac{\partial u}{\partial y}=\frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}$$

D.C. (1) partially w.r to y

$$\frac{\partial u}{\partial y}=\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}=\frac{3(y^2-xz)}{x^3+y^3+z^3-3xyz}$$

D.C. (1) partially w.r. to z

$$\frac{\partial u}{\partial z}=\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}=\frac{3(z^2-xy)}{(x^3+y^3+z^3-3xyz)}$$

Scanned with CamScanner

52 Now

$$L.H.S. = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u\right] = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}+\frac{3(y^2-zx)}{x^3+y^3+z^3-3xyz}+\frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}\right]$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}\right]$$

$$= 3\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\frac{(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}\right]$$

$$= 3\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{1}{x+y+z}\right)$$

$$= 3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$= 3\left[(-)\frac{1}{(x+y+z)^2}+\frac{(-)1}{(x+y+z)^2}+\frac{1}{(x+y+z)^2}\right]$$

$$= 3\times\frac{(-)3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

Proved

Scanned with CamScanner

(4) If $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$

Ans → Given that $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$

or $\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

or $\cos u = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)} = \frac{\sqrt{x}\left(1+\frac{y}{x}\right)}{1+\sqrt{\frac{y}{x}}} = V$ (Let) ... (1)

Here V is a homogeneous function of two independent variables x and y and of degree $\frac{1}{2}$.

By Euler's theorem

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = nV = \frac{1}{2}V \quad \cdots (2)$$

Again from equation (1) $V = \cos u$

$$\therefore \frac{\partial V}{\partial x} = -\sin u \frac{\partial u}{\partial x}$$

& $$\frac{\partial V}{\partial y} = -\sin u \frac{\partial u}{\partial y}$$

Now from equation (2)

$$(-)\ x\cdot\sin u\frac{\partial u}{\partial x} - y\cdot\sin u\frac{\partial u}{\partial y} = \frac{1}{2}V$$

or $$-\sin u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{1}{2}V = \frac{1}{2}\cos u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

Scanned with CamScanner

(4) If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

Ans, Given that

$$u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$\text{or } \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\text{or } \sin u = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)} = \frac{\sqrt{x}\left(1+\frac{y}{x}\right)}{1+\sqrt{\frac{y}{x}}} = V \text{ (Let)}$$

Here V is a homogeneous function of two independent variables x and y of degree $\frac{1}{2}$.

By Euler's theorem

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = nV = \frac{1}{2}V$$

$$\text{or } x\frac{\partial}{\partial x}(V) + y\frac{\partial}{\partial y}(V) = \frac{1}{2}V$$

$$\text{or } x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

$$\text{or } x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$\text{or } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u} = \frac{1}{2}\tan u$$

$$\text{or } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

Scanned with CamScanner

(*) If $u = \sin^{-1}\frac{x^2+y^2}{x+y}$

Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

Ans Given that

$$u = \sin^{-1}\frac{x^2+y^2}{x+y}$$

$$\text{or } \sin u = \frac{x^2+y^2}{x+y} = \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x\left(1+\frac{y}{x}\right)}$$

$$\text{or } \sin u = x\,\phi\left(\frac{y}{x}\right) = v \text{ (let)} \quad \cdots (1)$$

v is a homogeneous function of degree 1.

By Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 1\cdot v = v \quad \cdots (2)$$

$\because v = \sin u$

$$\therefore \frac{\partial v}{\partial x} = \cos u\frac{\partial u}{\partial x} \text{ \& } \frac{\partial v}{\partial y} = \cos u\frac{\partial u}{\partial y}$$

From (2) $x\cdot\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = v = \sin u$

$$\text{or } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u} = \tan u$$

(*) If $v = \tan^{-1}\frac{x^3+y^3}{x-y}$, then Show that

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \sin 2v.$$

Ans Given that $v = \tan^{-1}\frac{x^3+y^3}{x-y}$

$$\text{or } \tan v = \frac{x^3+y^3}{x-y}$$

Scanned with CamScanner

$$\text{or } \tan V = \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(1-\frac{y}{x}\right)} = \frac{x^2\left[1+\left(\frac{y}{x}\right)^3\right]}{\left(1-\frac{y}{x}\right)}$$

D.C.56

$$\text{or } \tan V = u \text{ (say)} \quad \cdots\cdots (1)$$

u is a homogeneous function of two independent variables x and y of degree 2.

By Eulir's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 2u \quad \cdots (2)$$

From (1) $\quad u = \tan V$

$$\therefore \frac{\partial u}{\partial x} = \sec^2 V \frac{\partial V}{\partial x} \quad \& \quad \frac{\partial u}{\partial y} = \sec^2 V \frac{\partial V}{\partial y}$$

Now from (1) $\quad x \cdot \sec^2 V \frac{\partial V}{\partial x} + y \sec^2 V \cdot \frac{\partial V}{\partial x} = 2u = 2\tan V$

$$\text{or } x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial x} = 2\tan V \cdot \frac{1}{\sec^2 V}$$

$$= 2\frac{\sin V}{\cos V} \cdot \cos^2 V$$

$$= 2\sin V \cdot \cos V$$

$$= \sin 2V$$

$$\text{or } x \cdot \frac{\partial V}{\partial x} + y\frac{\partial V}{\partial x} = \sin 2V$$

(*) Verify Euler's theorem when $u = x^3 \log\frac{y}{x}$

Ans, Given that $u = x^3 \log\frac{y}{x}$

$$\text{or } u = x^3(\log y - \log x)$$

$$\text{Now } \frac{\partial u}{\partial x} = 3x^2(\log y - \log x) - x^3 \cdot \frac{1}{x}$$

Scanned with CamScanner

or $x\frac{\partial u}{\partial x} = 3x^3(\log y - \log x) - x^3 \cdots (1)$

Again $\frac{\partial u}{\partial y} = x^3 \cdot \frac{1}{y}$

or $y\frac{\partial u}{\partial y} = x^3 \cdots (2)$

Adding (1) & (2) we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3x^3(\log y - \log x) - x^3 + x^3$$
$$= 3x^3(\log y - \log x)$$
$$= 3x^3 \log\frac{y}{x}$$
$$= 3u \quad (\text{let})$$
$$u = x^3 \log\left(\frac{y}{x}\right) = x^3\phi\left(\frac{y}{x}\right)$$

Here u is a homogeneous function of degree 3.

Therefore we find that the given function satisfies Euler's theorem.

(*) Verify Euler's theorem for the expression $x^n \sin\frac{y}{x}$.

Ans → Let $u = x^n \sin\frac{y}{x} \cdots (1)$

D.C. partially w.r. to x

$$\frac{\partial u}{\partial x} = nx^{n-1}\sin\frac{y}{x} - x^n \cos\frac{y}{x}\,\frac{y}{x^2}$$

or $x\frac{\partial u}{\partial x} = nx^n \sin\frac{y}{x} - x^{n-1}\cos\frac{y}{x}\cdot y \cdots (2)$

D.C. (1) partially w.r. to y

$$\frac{\partial u}{\partial y} = x^n \cos\frac{y}{x}\cdot\frac{1}{x}$$

Scanned with CamScanner

or $y\frac{\partial u}{\partial y} = y \cdot x^{n-1}\cos\frac{y}{x}$ ---(3)

Adding Equation (2) & (3) we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n \sin\frac{y}{x} - x^{n-1}\cos\frac{y}{x}\cdot y + y\cdot x^{n-1}\cos\frac{y}{x}$$

$$= nx^n \sin\frac{y}{x}$$

or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n \sin\frac{y}{x} = nu$

So the given function $x^n \sin\frac{y}{x}$ is a homogeneous function of degree n. Thus the given function satisfies Euler's theorem.

(4) If the sides of a plane triangle ABC vary in such a way that its circum radius remains constant prove that

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0$$

Ans → In $\triangle ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

or $a = 2R\sin A$

$b = 2R\sin B$

$c = 2R\sin C$

Differentiating

$\delta a = 2R\cos A\,\delta A$

or $\frac{\delta a}{\cos A} = 2R\,\delta A$

Scanned with CamScanner

Similarly $\frac{\delta b}{\cos B} = 2R\,\delta B$

$\frac{\delta c}{\cos C} = 2R\,\delta C$

Adding them we get

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 2R(\delta A + \delta B + \delta C) \quad \ldots\ldots (1)$$

we know that in any triangle ABC

$$A + B + C = \pi$$

Differentiating

$$\delta A + \delta B + \delta C = 0$$

Now from equation (1)

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 2R \cdot 0 = 0$$

Scanned with CamScanner