Sufficient Condition

Given that

$a \in H, b \in H \Rightarrow ab^{-1} \in H$ ----(1)

we have to prove that H is a subgroup of the group G. In other words it should satisfy all the conditions of a group.

Let e be the identity element of G.

Put $b = a$

So $a \in H, a \in H \Rightarrow aa^{-1} \in H$

$\Rightarrow e \in H$

Thus the identity element exists in H.

Again $e \in H$ and $a \in H \Rightarrow ea^{-1}$ i.e. $a^{-1} \in H$

So the inverse axiom is satisfied.

Again $b \in H$ and $b^{-1} \in H$

So by (1) $a \in H, b^{-1} \in H \Rightarrow a(b^{-1})^{-1} \in H$

$\Rightarrow ab \in H$

Thus the closure axiom is satisfied.

Since Composition in G is associative, G is associative, G being a group.

So Composition in H is also associative.

Hence all the Conditions are satisfied and we conclude that

H is a subgroup.

Thus the Sufficient Condition is satisfied.

424 The intersection of any two subgroups of a group is again a Subgroup of that group.

(or)

If H_1 and H_2 be any two subgroups of a group G, then prove that $H_1 \cap H_2$ is also a subgroup of G.

Proof) Let H_1 and H_2 be any two subgroups and we have to prove that $H_1 \cap H_2$ is a subgroup.

Now, $a, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

$\because a \in H_1 \cap H_2 \Rightarrow a \in H_1, a \in H_2$

& $b \in H_1 \cap H_2 \Rightarrow b \in H_1, b \in H_2$

Again $a \in H_1, b \in H_1 \Rightarrow ab^{-1} \in H_1$ ($\because H_1$ is a subgroup)

$a \in H_2, b \in H_2 \Rightarrow ab^{-1} \in H_2$ ($\because H_2$ is a subgroup)

$\therefore ab^{-1} \in H_1$

& $ab^{-1} \in H_2$

This concludes $ab^{-1} \in H_1 \cap H_2$

Hence $H_1 \cap H_2$ is a subgroup.

425 * Every Subgroup H of a cyclic group G is also a cyclic group.

Proof → Let H be a subgroup of cyclic group G which is generated by a. So every member of H will be evidently some positive or negative powers of a.

Let $a^m \in H$, where m is the smallest positive integer.

Let $a^k \in H$

So $k > m$

By the division algorithm

$k = qm + r$, where $0 \leq r < m$

Hence $a^k = a^{qm+r} = a^{qm} \cdot a^r = (a^m)^q \cdot a^r$

or $a^r = \dfrac{a^k}{(a^m)^q} = a^k \cdot (a^m)^{-q}$

Since $a^m \in H$ & $a^k \in H$

So $a^r \in H$

But m is the smallest positive integer such that $a^m \in H$.

Since $r < m$, we have $r = 0$

So $k = qm + r = qm + 0 = qm$

Hence every element a^k of H is of the form $(a^m)^q$ for some integer k.

This shows that H is a cyclic group generated by a^m.

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426

Define left coset and right coset.

Let $H = \{h_1, h_2, \ldots h_n\}$ be a subgroup of group G and a be any element of G then the set $\{h_1 a, h_2 a \ldots h_n a\}$ is called the right coset. It is denoted by Ha.

Also the set $\{a h_1, a h_2, \ldots a h_n\}$ is called the left coset. It is denoted by aH.

* If H is a subgroup of a group G and a is any arbitrary element of G prove that

$$Ha = H \Leftrightarrow a \in H$$
$$aH = H \Leftrightarrow a \in H$$

Proof, Let $Ha = H$ ... (1)

Then to prove that $a \in H$

we have $e \in Ha \in H \Rightarrow ea \in Ha$

or $a \in Ha$

or $a \in H$ (by (1))

Proved

Again to prove $Ha = H$

we have $a \in H$, Let $x \in Ha$

$\Rightarrow x = ha$ where $h \in H$ and $a \in H$

Although H is a subgroup

so $h \in H, a \in H \Rightarrow ha \in H$ ... (2)

Hence $x \in Ha \Rightarrow x = ha \in H$

$\therefore Ha \subset H \quad \cdots (2)$

Again let $y \in H$ then

$y = y a^{-1} a \quad [\because a^{-1}a = e]$

or $y = (y a^{-1}) a$

As $a \in H \Rightarrow a^{-1} \in H$ [H is a subgroup]

Also $y \in H, a^{-1} \in H \Rightarrow y a^{-1} \in H$

$\therefore y (y a^{-1}) a \in Ha$

$\therefore H \subset Ha \quad \cdots (3)$

From (2) & (3) we conclude that

$H = Ha$

Similarly we can prove that $aH = H \Leftrightarrow a \in H$

☆ If a, b are any two distinct elements of G then

$Ha = Hb \Longleftrightarrow ab^{-1} \in H$

$aH = bH \Longleftrightarrow ab^{-1} \in H$

Proof, Let $Ha = Hb$

$\therefore Hab^{-1} = Hbb^{-1} = He = H$

But $Hh = H$, when $h \in H$

{Because we know that left and right cosets corresponding to any element $h \in H$ are H. i.e. $hH = Hh = H$}

Hence $ab \in H$

Converse → $ab^{-1} \in H$

To prove $Ha = Hb$

Since $ab^{-1} \in H$

$\therefore Hab^{-1} = H \quad [\because Hh = H, \text{ when } h \in H]$

$\therefore Hab^{-1}b = Hb$

or $Ha = Hb$ Proved

Again, $aH = bH$

$\therefore b^{-1}aH = b^{-1}bH = eH = H$

but $b = H$, when $h \in H$

$\therefore b^{-1}a \in H$

Conversely, $b^{-1}a \in H$

To prove that $aH = bH$

Since $b^{-1}aH = H \quad \because hH = H$ when $h \in H$

or $bb^{-1}aH = bH$

or $aH = bH$ Proved

Any two right (left) cosets of a subgroup H of G are either disjoint or identical.

Proof, Let H be a subgroup of group G.

where $a, b \in G$

$H = \{h_1, h_2, h_3 \ldots\}$

then $Ha = \{h_1a, h_2a, h_3a \ldots\}$

$Hb = \{h_1b, h_2b, h_3b, \ldots\}$

are two right cosets.

Suppose that ~~the~~ two right cosets Ha and Hb have some common elements.

429 Now we will show that all the elements in Ha and Hb are common.

This will establish the fact that they are either identical or disjoint

i.e. either $Ha \cap Hb = \phi$

i.e. $Ha = Hb$

Let $h_1 a = h_2 b$ be the common elements of two cosets.

$\therefore h_2^{-1} h_1 a = h_2^{-1} h_2 b = eb = b$

Now $h_1 \in H$, $h_2^{-1} \in H$

$\therefore h_2^{-1} h_1 \in H$ as it is a subgroup.

Again Let $h_2^{-1} h_1 = h_i$ (suppose that)

$\therefore h_i a = b$

Taking cosets of both sides

$\therefore H(h_i a) = Hb$

or $(H h_i) a = Hb$ .... (1)

But $h_i \in H$ [$\because$ Left and right cosets corresponding to any element $h \in H$ are H i.e. $hH = Hh = H$]

$\therefore H h_i = H$

$\therefore Ha = Hb$ [from (1)]

Hence the two right cosets are identical if they have some elements in common or else they are adjoint.

Similarly by taking $ah_1 = bh_1$, we get

$a = bh_2 h_1^{-1} = bh_i$ (let)

or $aH = (bh_i)H = b(h_i H) = bH$

$\because h_i H = H$, where $h_i \in H$

Thus we can also prove that the left cosets are either identical or disjoint.