LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER WITH VARIABLE CO-EFFICIENTS

The standard form of differential equation of Second order with Variable Co-efficient is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \cdots (1)$$

where P, Q are functions of x and R is constant or function of x.

Method of solution of differential equation of the given equation like equation (1).

Here we shall discuss certain methods to solve the diff. eqn.

Method (1) → The Complete Solution in the terms of a known Complementry function or integral.

The given differential equation is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \ldots (1)$$

Let $y = u$ be a known integral in the C.F. of the equation (1).

i.e. $y = u$ is the solution of equation.

$$\therefore \frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qu = 0 \quad \ldots (2)$$

Let $y = uv$ be C.F. of (1)

Put $y = uv$

D.c. w.r.to n

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Again d.c. w.r.to x

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + \frac{du}{dx}\cdot\frac{dv}{dx} + v\frac{d^2u}{dx^2} + \frac{dv}{dx}\cdot\frac{du}{dx}$$

$$= v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + u\cdot\frac{d^2v}{dx^2}.$$

Now From equation (1)

$$v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + u\cdot\frac{d^2v}{dx^2} + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Quv = R$$

$$\text{or } u\cdot\frac{d^2v}{dx^2} + \frac{dv}{dx}\left(2\frac{du}{dx} + Pu\right) + v\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Q\cdot u\right) = R$$

or, $u\cdot\frac{d^2v}{dx^2}+\frac{dv}{dx}\left(2\frac{du}{dx}+Pu\right)$
$+v\cdot 0=R$, [where $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qu=0$ from equ^n (2)]

or $$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}\quad\cdots(3)$$

put $\frac{dv}{dx}=z$

So, $\frac{d^2v}{dx^2}=\frac{dz}{dx}$

Now from equation (3)
$$\frac{dz}{dx}+\left(P+\frac{2}{u}\frac{du}{dx}\right)z=\frac{R}{u}\quad\cdots(4)$$
which is linear with z as a dependent variable.

$$I\cdot F=e^{\int\left(P+\frac{2}{u}\frac{du}{dx}\right)dx}$$
$$=e^{\int\left(P\,dx+\frac{2}{u}du\right)}=e^{\int P\,dx}\cdot e^{2\int\frac{du}{u}}$$
$$=e^{2\log u}\cdot e^{\int P\,dx}=e^{\log u^2}\cdot e^{\int P\,dx}$$
$$=u^2\cdot e^{\int P\,dx}$$

Hence Solution of equ^n (4) is
$$z\cdot u^2\cdot e^{\int P\,dx}=\int\left[\frac{R}{u}\cdot u^2\cdot e^{\int P\,dx}\right]dx+k\quad\cdots(5)$$

or $z \cdot u^2 \cdot e^{\int P\,dx} = \int R \cdot u \cdot e^{\int P\,dx}\,dx + K$

or $z = \dfrac{1}{u^2 \cdot e^{\int P\,dx}} \cdot \int R \cdot u \cdot e^{\int P\,dx} \cdot dx + K \cdot \dfrac{1}{u^2 \cdot e^{\int P\,dx}}$

or $\dfrac{dv}{dx} = \dfrac{e^{-\int P\,dx}}{u^2} \int u \cdot R \cdot e^{\int P\,dx}\,dx + \dfrac{K}{u^2}\, e^{-\int P\,dx}$

Integrating

$$v = \int \left[\frac{e^{-\int P\,dx}}{u^2} \int u \cdot R\, e^{\int P\,dx}\,dx \right] dx + K \int \frac{e^{-\int P\,dx}}{u^2}\,dx + K_2$$

$\because y = uv$

Hence ~~the~~ solution of equation (1), is

$$y = K_2 \cdot u + u \int \left[\frac{e^{-\int P\,dx}}{u^2} \int u \cdot R\, e^{\int P\,dx}\,dx \right] dx + k \cdot u \cdot \int \frac{e^{-\int P\,dx}}{u^2}\,dx \quad \ldots (6)$$

Equation (6) Contains the given Solution $y = u$ and it is Complete primitive or general solution of equation (1).

It is evident from eqn (6) that

C.F is $u\displaystyle\int \frac{e^{-\int P\,dx}}{u^2}\,dx$

and the

P.I is

$$u\int\left[\frac{e^{-\int P\,dx}}{u^2}\int u\cdot R\,e^{\int P\,dx}\,dx\right]dx$$