21 (*) Solve 21

$$(x\sin x+\cos x)\frac{d^2y}{dx^2}-x\cos x\frac{dy}{dx}+y\cos x = \sin x\,(x\sin x+\cos x)^2$$

Ans, Given that

$$(x\sin x+\cos x)\frac{d^2y}{dx^2}-x\cos x\frac{dy}{dx}+y\cos x = \sin x\,(x\sin x+\cos x)^2$$

or $$\frac{d^2y}{dx^2}-\frac{x\cos x}{x\sin x+\cos x}\frac{dy}{dx}+\frac{\cos x}{x\sin x+\cos x}y=\sin x\,(x\sin x+\cos x) \quad \text{(1)}$$

Comparing it with standard form

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R,$$

we have $$P=\frac{(-)\,x\cos x}{x\sin x+\cos x}$$

$$Q=\frac{\cos x}{x\sin x+\cos x},\quad R=\sin x\,(x\sin x+\cos x)^2$$

Here we find that

$$P+Qx=\frac{-x\cos x}{x\sin x+\cos x}+\frac{x\cdot\cos x}{x\sin x+\cos x}=0$$

So, $y=x$ is a part of the C.F. of the solution of equation (1)

Now $y=vx$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

So, $$\frac{d^2y}{dx^2}=\frac{dv}{dx}+\frac{dv}{dx}+x\frac{d^2v}{dx^2}=x\cdot\frac{d^2v}{dx^2}+2\frac{dv}{dx}$$

Now from equation (1) we get

$$x\frac{d^2v}{dx^2}+2\frac{dv}{dx}-\frac{x\cos x}{x\sin x+\cos x}\left[v+x\frac{dv}{dx}\right]+\frac{\cos x}{x\sin x+\cos x}\cdot vx=\sin x\,(x\sin x+\cos x)$$

or $$\frac{d^2v}{dx^2}+\frac{dv}{dx}\left[\frac{2}{x}-\frac{x\cos x}{x\sin x+\cos x}\right]-\cancel{\frac{\cos x}{x\sin x+\cos x}\cdot v}+\cancel{\frac{\cos x}{x\sin x+\cos x}\cdot v}=\frac{\sin x\,(x\sin x+\cos x)}{x}$$

or $$\frac{d^2v}{dx^2}+\frac{dv}{dx}\left[\frac{2}{x}-\frac{x\cos x}{x\sin x+\cos x}\right]=\frac{\sin x\,(x\sin x+\cos x)}{x}$$

where $p=\frac{dv}{dx}$

$$\frac{dp}{dx}=\frac{d^2v}{dx^2}$$

or $$\frac{dp}{dx}+p\left(\frac{2}{x}-\frac{x\cos x}{x\sin x+\cos x}\right)=\frac{\sin x\,(x\sin x+\cos x)}{x}$$

which is linear

$$I.F.=e^{\int\left(\frac{2}{x}-\frac{x\cos x}{x\sin x+\cos x}\right)dx}$$

$$=e^{2\int\frac{dx}{x}-\int\frac{x\cos x\,dx}{x\sin x+\cos x}}$$

$$=e^{2\log x-\log(x\sin x+\cos x)}$$

$$=e^{\log\frac{x^2}{x\sin x+\cos x}}=\frac{x^2}{x\sin x+\cos x}$$

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$$\therefore\ \text{p.}\ \frac{x^2}{x\sin x+\cos x} = \int \frac{\sin x\,\cancel{(x\sin x+\cos x)}}{\cancel{x}} \cdot \frac{x^2}{\cancel{(x\sin x+\cos x)}}\,dx + k$$

$$= \int x\sin x\,dx + k$$

$$= x\int \sin x\,dx - \int\left\{\frac{d}{dx}\text{ of } x \int \sin x\,dx\right\}dx + k$$

$$= -x\cos x + \int \cos x + k$$

$$\text{or p.}\ \frac{x^2}{x\sin x+\cos x} = -x\cos x + \sin x + k$$

$$\text{or } p = \frac{du}{dx} = \frac{x\sin x+\cos x}{x^2}\cdot(-x\cos x+\sin x) + \frac{k}{x^2}(x\sin x+\cos x)$$

$$\text{or } \frac{du}{dx} = \frac{-x^2\sin x\cos x + x\sin^2 x - x\cos^2 x + \sin x\cos x}{x^2} + k\left(\frac{\sin x}{x}+\frac{\cos x}{x^2}\right)$$

$$= -\sin x\cos x - \frac{1}{x}(\cos^2 x - \sin^2 x) + \frac{1}{x^2}\sin x\cos x + k\left(\frac{\sin x}{x}+\frac{\cos x}{x^2}\right)$$

$$\text{or, } \frac{du}{dx} = \frac{(-1)}{2}\,2\sin x\cos x - \frac{1}{x}\cos 2x + \frac{1}{2x^2}\sin 2x + k\left(\frac{\sin x}{x}+\frac{\cos x}{x^2}\right)$$

$$\text{or } du = -\frac{1}{2}\sin 2x\,dx - \frac{1}{2}\left[\frac{2}{x}\cos 2x - \frac{1}{x^2}\sin 2x\right]dx + k\left[\frac{\sin x}{x}+\frac{\cos x}{x^2}\right]dx$$

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$$\int dv = -\frac{1}{2}\int \sin 2x\,dx - \frac{1}{2}\int\left[\frac{2}{x}\cos 2x - \frac{1}{x^2}\sin 2x\right]dx$$
$$+ k\int\left[-\frac{\sin x}{x} + \frac{\cos x}{x^2}\right]dx + k_1$$

or $v = \frac{1}{4}\cos 2x - \frac{1}{2}\,\frac{\sin 2x}{x} - k\cdot\frac{\cos x}{x} + k_1$

So the complete solution of equation (1) 2.5

$$y = vx$$

or $y = \frac{1}{4}x\cos 2x - \frac{1}{2}\sin 2x - k\cos x + x.k_1$

Solve $x^2\frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = x^3e^x$

Ans Given that

$$x^2\frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = x^3e^x$$

or $\frac{d^2y}{dx^2} - \left(1+\frac{2}{x}\right)\frac{dy}{dx} + \left(\frac{1}{x}+\frac{2}{x^2}\right)y = xe^x$ ...(1)

Here $P = (-)\left(1+\frac{2}{x}\right),\ Q = \frac{1}{x}+\frac{2}{x^2}$

& $R = xe^x$

Here $P + Qx = -1 - \frac{2}{x} + x\left(\frac{1}{x}+\frac{2}{x^2}\right)$

$= \cancel{-1} - \cancel{\frac{2}{x}} + \cancel{1} + \cancel{\frac{2}{x}}$

$= 0$

$\because P + Qx = 0$

So $y = x$ is a part of C.F.

Now put $y = vx$

D.C.

$$\frac{dy}{dx} = \frac{dv}{dx}\cdot x + v$$

and
$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}\cdot x + 2\frac{dv}{dx}$$

Now from equation (1)

$$\frac{d^2v}{dx^2}x + 2\frac{dv}{dx} - \left(1+\frac{2}{x}\right)\left(x\frac{dv}{dx}+v\right) + \left(\frac{1}{x}+\frac{2}{x^2}\right)vx = x\,e^x$$

or,
$$\frac{d^2v}{dx^2} + \frac{dv}{dx}\cdot\frac{2}{x} - \frac{2}{\cancel{x}}\cdot\cancel{x}\frac{dv}{dx}\cdot\frac{1}{x} - \cancel{x}\frac{dv}{dx}\cdot\frac{1}{\cancel{x}} - \frac{1}{x}\cdot v - \frac{1}{x}\cdot\frac{2}{x}\cdot v + \frac{1\cdot v}{x} + \frac{2v}{x^2} = e^x$$

or
$$\frac{d^2v}{dx^2} + \frac{dv}{dx}\left[\frac{\cancel{2}}{\cancel{x}} - \frac{\cancel{2}}{\cancel{x}} - 1\right] - \frac{\cancel{v}}{\cancel{x}} - \frac{\cancel{2v}}{\cancel{x^2}} + \frac{\cancel{v}}{\cancel{x}} + \frac{\cancel{2v}}{\cancel{x^2}} = e^x$$

or
$$\frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x$$

Now $p = \dfrac{dv}{dx}$ So $\dfrac{dp}{dx} = \dfrac{d^2v}{dx^2}$

or
$$\frac{dp}{dx} - p = e^x$$

which is linear

$I.F. = e^{\int(-)dx} = e^{-x}$

$\therefore p \cdot e^{-x} = \int e^{x} \cdot e^{-x}\, dx + k$

$p \cdot e^{-x} = \int dx + k = x + k_1$

or $p = x \cdot e^{x} + k e^{x}$

or $\frac{du}{dx} = x e^{x} + k e^{x}$

or $\int du = \int x e^{x}\, dx + k \int e^{x}\, dx$

Integrating

$u = x e^{x} - \int\{\text{d.c. of } x \int e^{x}\, dx\}\, dx + k e^{x} + k_1$

$u = x e^{x} - e^{x} + k e^{x} + k_1$

The Complete Solution of the equation (1) ~~2's~~

$y = ux$

$= x^2 e^{x} - x e^{x} + k x e^{x} + k_1 x$