PARTIAL DIFFERENTIATION

INTRODUCTION

Function to be partially differentiated is always considered as the function of a single variable, with respect to which the partial differentiation is to be done, at the same time all other variables are treated as constants.

Let $u = f(x, y)$ be a function of two variables x and y.

The partial derivative of u with respect to x is denoted by $\frac{\partial u}{\partial x}$ or u_x or f_x.

It means that u has been differentiated partially w.r. to x only and y has been treated as constant. [we call $\partial \to$ del]

So if $u = f(x, y)$ then

$$\frac{\partial u}{\partial x} = \underset{h \to 0}{Lt} \frac{f(x+h, y) - f(x, y)}{h}$$

if the limit exists.

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Similarly, if $u = f(x, y)$ then

$$\frac{\partial u}{\partial y} = \underset{k \to 0}{Lt} \frac{f(x, y+k) - f(x, y)}{k}$$

if this limit exists.

Euler's theorem

State and prove Euler's theorem on partial differentiation of homogeneous function of two independent variables.

Statement → If $f(x, y)$ be a homogeneous function of degree n then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

Proof → Let

$$f(x, y) = A x^{\alpha_1} y^{\beta_1} + B x^{\alpha_2} y^{\beta_2} + C x^{\alpha_3} y^{\beta_3} + \cdots \quad (1)$$

where $\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = \alpha_3 + \beta_3 = \cdots = n \quad \cdots (2)$

and $A, B, C, \cdots$ are constants.

D.C. (1) partially w.r. to x keeping y constant

$$\frac{\partial f}{\partial x} = A\alpha_1 x^{\alpha_1 - 1} y^{\beta_1} + B\alpha_2 x^{\alpha_2 - 1} y^{\beta_2} + C\alpha_3 x^{\alpha_3 - 1} y^{\beta_3} + \cdots$$

or $$x \frac{\partial f}{\partial x} = A\alpha_1 x^{\alpha_1} y^{\beta_1} + B\alpha_2 x^{\alpha_2} y^{\beta_2} + C\alpha_3 x^{\alpha_3} y^{\beta_3} + \cdots \quad (3)$$

Again differentiating equⁿ (1) partially w.r. to y Keeping x constant.

$$\frac{\partial f}{\partial y} = A\beta_1 x^{\alpha_1} y^{\beta_1 - 1} + B\beta_2 x^{\alpha_2} y^{\beta_2 - 1} + C\beta_3 x^{\alpha_3} y^{\beta_3 - 1} + \cdots$$

or $$y\frac{\partial f}{\partial y} = A\beta_1 x^{\alpha_1} y^{\beta_1} + B\beta_2 x^{\alpha_2} y^{\beta_2} + C\beta_3 x^{\alpha_3} y^{\beta_3} + \cdots \quad (4)$$

Adding equation (3) and (4) we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = A\alpha_1 x^{\alpha_1} y^{\beta_1} + B\alpha_2 x^{\alpha_2} y^{\beta_2} + C\alpha_3 x^{\alpha_3} y^{\beta_3} + \cdots + A\beta_1 x^{\alpha_1} y^{\beta_1} + B\beta_2 x^{\alpha_2} y^{\beta_2} + C\beta_3 x^{\alpha_3} y^{\beta_3} + \cdots$$

$$= Ax^{\alpha_1} y^{\beta_1}(\alpha_1 + \beta_1) + Bx^{\alpha_2} y^{\beta_2}(\alpha_2 + \beta_2) + Cx^{\alpha_3} y^{\beta_3}(\alpha_3 + \beta_3) + \cdots$$

$$= Ax^{\alpha_1} y^{\beta_1} \cdot n + Bx^{\alpha_2} y^{\beta_2} \cdot n + Cx^{\alpha_3} y^{\beta_3} \cdot n + \cdots$$ [using equⁿ (2)]

$$= n\left[Ax^{\alpha_1} y^{\beta_1} + Bx^{\alpha_2} y^{\beta_2} + Cx^{\alpha_3} y^{\beta_3} + \cdots\right]$$

$$= nf$$ [from equⁿ (1)]

or $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Hence proved.

④ If $u = \dfrac{x^2+y^2}{x+y}$, prove that

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left\{1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right\}$$

Ans, Given that

$$u = \frac{x^2+y^2}{x+y} \quad \ldots\ldots (1)$$

D.C. partially w.r. to x, keeping y constant.

$$\frac{\partial u}{\partial x} = \frac{(x+y)\cdot 2x - (x^2+y^2)\cdot 1}{(x+y)^2}$$

$$= \frac{2x^2+2xy-x^2-y^2}{(x^2+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

D.C. (1) partially w.r. to y, keeping x constant

$$\frac{\partial u}{\partial y} = \frac{2y(x+y)-(x^2+y^2)\cdot 1}{(x+y)^2}$$

$$= \frac{2xy+2y^2-x^2-y^2}{(x+y)^2} = \frac{y^2+2xy-x^2}{(x+y)^2}$$

Now $\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y} = \dfrac{x^2+\cancel{2xy}-y^2-y^2-\cancel{2xy}+x^2}{(x+y)^2}$

$$= \frac{2x^2-2y^2}{(x+y)^2} = \frac{2(x^2-y^2)}{(x+y)^2}$$

$$= \frac{2(x+y)(x-y)}{(x+y)^2}$$

or $\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y} = \dfrac{2(x-y)}{(x+y)}$

$$L.H.S. = \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2$$

$$= \frac{4(x-y)^2}{(x+y)^2}$$

Now, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$

$$= \frac{x^2+2xy-y^2}{(x+y)^2} + \frac{y^2+2xy-x^2}{(x+y)^2}$$

$$= \frac{\cancel{x^2}+2xy-\cancel{y^2}+\cancel{y^2}+2xy-\cancel{x^2}}{(x+y)^2}$$

$$= \frac{4xy}{(x+y)^2}$$

$$R.H.S. = 4\left\{1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right\} = 4\left\{1 - \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)\right\}$$

$$= 4\left\{1 - \frac{4xy}{(x+y)^2}\right\}$$

$$= 4 \cdot \frac{(x+y)^2 - 4xy}{(x+y)^2}$$

$$= 4 \cdot \frac{(x-y)^2}{(x+y)^2}$$

$$= L.H.S.$$

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Ans, Given that

$$u = \frac{x^2 y^2}{x+y} \quad \ldots\ldots (1)$$

D.C. (1) partially w.r. to x, y as Constant

$$\frac{\partial u}{\partial x} = \frac{2xy^2(x+y) - x^2y^2 \cdot 1}{(x+y)^2}$$

$$= \frac{2xy^2}{x+y} - \frac{x^2y^2}{(x+y)^2}$$

$$\frac{\partial u}{\partial x} = y^2\left[\frac{2x}{x+y} - \frac{x^2}{(x+y)^2}\right]$$

or $$x \cdot \frac{\partial u}{\partial x} = \frac{2x^2y^2}{x+y} - \frac{x^3y^2}{(x+y)^2} \quad \ldots (2)$$

D.C. (1) partially w. r. to y, keeping x Constant.

$$\frac{\partial u}{\partial y} = \frac{(x+y)\,2yx^2 - x^2y^2 \cdot 1}{(x+y)^2}$$

$$= \frac{2yx^2}{(x+y)^2} - \frac{x^2y^2}{(x+y)^2}$$

or $$y\frac{\partial u}{\partial y} = \frac{2y^2x^2}{(x+y)^2} - \frac{x^2y^3}{(x+y)^2} \quad \ldots (3)$$

Adding equation (2) & (3) we get

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45 $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2x^2y^2}{x+y} - \frac{x^3y^2}{(x+y)^2}$ D.C-45

$$+ \frac{2x^2y^2}{x+y} - \frac{x^2y^3}{(x+y)^2}$$

$$= \frac{4x^2y^2}{x+y} - \left[\frac{x^3y^2 + x^2y^3}{(x+y)^2}\right]$$

$$= \frac{4x^2y^2}{x+y} - \frac{x^2y^2\cancel{(x+y)}}{(x+y)^{\cancel{2}}}$$

$$= \frac{4x^2y^2 - x^2y^2}{x+y} = \frac{3x^2y^2}{x+y}$$

$$= 3u \quad [\text{frm equn } (1)]$$

Or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$ Proved

(7) Show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

when $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

Ans, Given that

$$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) \quad \cdots (1)$$

D.C. partially w.r.to x

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot -\frac{1}{x^2}\cdot y$$

$$= \frac{\cancel{y}}{\sqrt{y^2-x^2}} \cdot \frac{1}{\cancel{y}} + \frac{\cancel{x^2}}{x^2+y^2} (-) \frac{y}{\cancel{x^2}}$$

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$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

Again differentiating equⁿ ① partially w.r. to y

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{(-)1}{y^2} \cdot x + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{(-)x}{y^{\cancel{2}}} + \frac{x^2}{x^2+y^2} \cdot \frac{1}{\cancel{x}}$$

$$= \frac{(-)x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$L.H.S. = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= x \cdot \frac{1}{\sqrt{y^2-x^2}} - \cancel{\frac{xy}{x^2+y^2}} - \frac{x\cancel{y}}{\cancel{y}\sqrt{y^2-x^2}} + \cancel{\frac{xy}{x^2+y^2}}$$

$$= \frac{x}{\sqrt{y^2-x^2}} - \frac{x}{\sqrt{y^2-x^2}} = 0$$

(4) If $u = (x^2+y^2+z^2)^{-\frac{1}{2}}$

Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u$

Ans, Given that

$$u = (x^2+y^2+z^2)^{-\frac{1}{2}} \quad \ldots\ldots ①$$

D.C. partially w.r. to x

$$\frac{\partial u}{\partial x} = (-)\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot 2x$$

$$= -x\,(x^2+y^2+z^2)^{-\frac{3}{2}}$$

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$$\frac{\partial u}{\partial y} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot 2y$$

$$= -y\,(x^2+y^2+z^2)^{-\frac{3}{2}}$$

D.C. (1) partially w.r. to z

$$\frac{\partial u}{\partial z} = -z\,(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$L.H.S = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

$$= (-)x^2(x^2+y^2+z^2)^{-\frac{3}{2}} + (-y^2)(x^2+y^2+z^2)^{-\frac{3}{2}} + (-z^2)(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= (-)(x^2+y^2+z^2)^{-\frac{3}{2}}(x^2+y^2+z^2)$$

$$= (-)(x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$= -u$$