Q) Solve $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = -4x^5$

given that $y = e^{x^2}$ is a solution if the left hand side is equated to zero.

Ans → Given that $y = e^{x^2}$

Putting $y = v \cdot e^{x^2}$ we have

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$$\frac{dy}{dx} = \frac{dv}{dx} e^{x^2} + v \cdot e^{x^2} \cdot 2x$$

$$\text{or, } \frac{dy}{dx} = \frac{dv}{dx} e^{x^2} + 2vx\, e^{x^2}$$

D.C.

$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} e^{x^2} + \frac{dv}{dx} \cdot e^{x^2} \cdot 2x + 2v \cdot x \cdot e^{x^2} \cdot 2x$$

$$+ 2v e^{x^2} + 2x e^{x^2} \frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} \cdot e^{x^2} + 4x e^{x^2} \frac{dv}{dx} + 4x^2 e^{x^2} \cdot v + 2e^{x^2} \cdot v \quad \cdots (1)$$

Now, from the given equation

$$x \frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3 y = -4x^5$$

$$\text{or } \frac{d^2y}{dx^2} - \frac{1}{x} \frac{dy}{dx} - 4x^2 y = -4x^4$$

Now from (1)

$$\frac{d^2v}{dx^2} e^{x^2} + 4x e^{x^2} \frac{dv}{dx} + 4x^2 e^{x^2} \cdot v + 2e^{x^2} \cdot v$$
$$- \frac{1}{x} \left[\frac{dv}{dx} e^{x^2} + 2vx\, e^{x^2} \right] - 4x^2 \cdot v \cdot e^{x^2} = -4x^4$$

$$\text{or } \frac{d^2v}{dx^2} e^{x^2} + 4x \cdot e^{x^2} \cdot \frac{dv}{dx} + \cancel{4x^2 \cdot e^{x^2} \cdot v} + \cancel{2e^{x^2} \cdot v}$$
$$- \frac{dv}{dx} \cdot \frac{1}{x} e^{x^2} - \cancel{2v \cdot e^{x^2}} - \cancel{4x^2 \cdot e^{x^2} \cdot v} = -4x^4$$

$$\text{or, } \frac{d^2v}{dx^2} \cdot e^{x^2} + \frac{dv}{dx} \left(4x - \frac{1}{x} \right) e^{x^2} = -4x^4$$

$$= -4x^4 \cdot e^{-x^2}$$

$$\text{or } \frac{d^2v}{dx^2} + \left(4x - \frac{1}{x} \right) \frac{dv}{dx}$$

or, $\frac{dp}{dx} + \left(4x - \frac{1}{x}\right)p = -4x^4 \cdot e^{-x^2}$

where $p = \frac{dv}{dx}$

which is linear

$$I.F = e^{\int\left(4x - \frac{1}{x}\right)dx} = e^{4\int x\,dx - \int \frac{dx}{x}}$$

$$= e^{4 \cdot \frac{x^2}{2} - \log x} = e^{2x^2 - \log x}$$

$$= e^{-\log x + 2x^2} = e^{\log \frac{1}{x}} \cdot e^{2x^2}$$

$$I.F. = \frac{1}{x} e^{2x^2}$$

$$\therefore p \cdot \frac{1}{x} \cdot e^{2x^2} = (-4)\int x^4 \cdot e^{-x^2} \cdot \frac{1}{x} \cdot e^{2x^2}\,dx + k$$

$$\text{or } p \cdot \frac{1}{x} \cdot e^{2x^2} = (-)4\int e^{x^2} \cdot x^3\,dx + k$$

$$\text{or } p \cdot \frac{1}{x} \cdot e^{2x^2} = (-)4\int e^{x^2} \cdot x^3\,dx + k \quad \left[\text{Put } x^2 = z,\ \text{D.C. } 2x\,dx = dz\right.$$

$$= (-4)\int e^z z \,\frac{dz}{2} + k$$

$$= \frac{1}{2}(-4)\left[z\int e^z dz - \int\left\{\frac{d}{dz} z \int e^z dz\right\}dz\right] + k$$

$$= (-2)\left[z e^z - \int e^z dz\right] + k$$

$$= (-2)\, e^z (z - 1) + k$$

$$\text{or } p \cdot \frac{1}{x} \cdot e^{2x^2} = (-2)\, e^{x^2}(x^2 - 1) + k$$

$$\text{or } p = \frac{du}{dx} = (-2)\, e^{-x^2}(x^2 \cancel{-x}) + Kx \cdot e^{-2x^2}$$

$$\text{or } du = (-2)\, e^{-x^2} \cdot x^3\, dx + 2\, e^{-x^2} \cdot x\, dx + k\, x\, e^{-2x^2} dx$$

Integrating

$$\int du = -2\int x^3 \cdot e^{-x^2} dx + 2\int x \cdot e^{-x^2} dx + k\int x\, e^{-2x^2} dx + k_1$$

$$\text{or } u = I_1 + I_2 + I_3 + k_1 \quad (\text{Let}) \text{ ---- } (2)$$

$$I_1 = -2\int x^3\, e^{-x^2}\, dx$$

Let $-x^2 = z$
D.C.
$-2x\,dx = dz$
$x\,dx = -\frac{dz}{2}$

$$= -2\int x^2 \cdot e^{-x^2} \cdot x\, dx$$

$$= \cancel{-2}\int (-z) e^z (-)\frac{dz}{\cancel{2}}$$

$$= (-)\int z\, e^z\, dz = (-)\, e^z(z-1) = e^z(1-z)$$

$$= e^{-x^2}(1+x^2)$$

$$I_2 = 2\int x\, e^{-x^2}\, dx$$

Let $-x^2 = z$
D.C.
$-2x\,dx = dz$
$x\,dx = (-)\frac{dz}{2}$

$$= \cancel{2}\int e^z (-)\frac{dz}{\cancel{2}}$$

$$= -e^z = -e^{-x^2}$$

$$I_3 = k\int x\, e^{-2x^2}\, dx$$

Let
$-2x^2 = z$
D.C.
$-4x\,dx = dz$
$\therefore x\,dx = (-)\frac{dz}{4}$

$$= k\int e^z (-)\frac{dz}{4} = -\frac{k}{4} e^z$$

$$= -\frac{k}{4}\, e^{-2x^2}$$

Now from equation (2)

$$v = e^{-x^2}(1+x^2) + (-)e^{-x^2} - \frac{k}{4} e^{-2x^2} + k_1$$

$$= \cancel{e^{-x^2}} + x^2 \cdot e^{-x^2} - \cancel{e^{-x^2}} - \frac{k}{4} e^{-2x^2} + k_1$$

$$v = x^2 \cdot e^{-x^2} - \frac{k}{4} e^{-2x^2} + k_1$$

So the complete solution of the given differential equation is

$$y = v e^{x^2}$$

$$\text{or } y = x^2 - \frac{k}{4} e^{-x^2} + k_1 e^{x^2}$$

(7) Solve by the method of variations of parameters

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x$$

Ans, The complementary function of the given equation i.e. the solution of the equation

$$\frac{d^2y}{dx^2} + 4y = 0 \quad \text{is}$$

$$y = k_1 \cos 2x + k_2 \sin 2x$$

where k_1 & k_2 are constants.

Let $y = A\cos 2x + B\sin 2x \quad \cdots (1)$ be the complete primitive of the given equation where A & B are functions of x, such that the equation is satisfied.

D.C. ① w.r.to x

$$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x + \frac{dA}{dx}\cos 2x + \frac{dB}{dx}\sin 2x \quad \cdots ②$$

Now we choose A & B such that

$$\frac{dA}{dx}\cos 2x + \frac{dB}{dx}\sin 2x = 0 \quad \cdots ③$$

Now from ②

$$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x$$

D.C. w.r.to x

$$\frac{d^2y}{dx^2} = -4A\cos 2x - 4B\sin 2x - 2\frac{dA}{dx}\sin 2x + 2\frac{dB}{dx}\cos 2x$$

Now putting the above values in

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x$$

we get

$$-4A\cos 2x - 4B\sin 2x - 2\frac{dA}{dx}\sin 2x + 2\frac{dB}{dx}\cos 2x + 4y = 4\tan 2x$$

or $-4(A\cos 2x + B\sin 2x) - 2\sin 2x\frac{dA}{dx} + 2\cos 2x\frac{dB}{dx} + 4y = 4\tan 2x$

or $\cancel{-4y} - 2\sin 2x\frac{dA}{dx} + 2\cos 2x\frac{dB}{dx} \cancel{+4y} = 4\tan 2x$ [using eqn ①]

or $-\sin 2x\frac{dA}{dx} + \cos 2x\frac{dB}{dx} = 2\tan 2x \quad \cdots ④$

From ③ & ④

we get

equ^n (3) $\longrightarrow$ $\frac{dA}{dx}\cos 2x + \frac{dB}{dx}\sin 2x = 0$

equ^n (4) $\longrightarrow$ $-\frac{dA}{dx}\sin 2x + \frac{dB}{dx}\cos 2x = 2\tan 2x$

multiplying equ^n (3) by $\sin 2x$ & (4) by $\cos 2x$ we get

$$\frac{dA}{dx}\sin 2x\cos 2x + \frac{dB}{dx}\sin^2 2x = 0$$

$$-\frac{dA}{dx}\sin 2x\cos 2x + \frac{dB}{dx}\cos^2 2x = \cos 2x \cdot 2\tan 2x$$

Adding $\frac{dB}{dx}(\sin^2 2x + \cos^2 2x) = \cancel{\cos 2x} \cdot \frac{2\sin 2x}{\cancel{\cos 2x}} = 2\sin 2x$

or $\frac{dB}{dx} = 2\sin 2x$

Now put the value of $\frac{dB}{dx}$ in equ^n (3)

$$\frac{dA}{dx}\cos 2x + 2\sin^2 2x = 0$$

or $\frac{dA}{dx}\cos 2x = -2\sin^2 2x$

or $\frac{dA}{dx} = -\frac{2\sin^2 2x}{\cos 2x}$

Now, we have,

$$\frac{dA}{dx} = -\frac{2\sin^2 2x}{\cos 2x}, \quad \frac{dB}{dx} = 2\sin 2x$$

Integrating we get

$$\int dA = -2\int \frac{\sin^2 2x}{\cos 2x}dx + K_1$$

38 $A = (-2)\int \frac{1-\cos^2 2x}{\cos 2x}\,dx + k_1$ 38

$= (-)2\int (\sec 2x - \cos 2x)\,dx + k_1$

$= -2\int \sec 2x + 2\int \cos 2x\,dx + k_1$

$A = -\log(\sec 2x + \tan 2x) + \sin 2x + k_1$

$\int dB = 2\int \sin 2x\,dx + k_2$

or $B = -\cos 2x + k_2$

Now putting the values of A and B in equation (1)

$y = [(-)\log(\sec 2x + \tan 2x)]\cos 2x + \cancel{\sin 2x}\cos 2x$
$+ k_1 \cos 2x - \sin 2x\,\cancel{\cos 2x} + k_2 \sin 2x$

or $y = k_1 \cos 2x + k_2 \sin 2x - [\log(\sec 2x + \tan 2x)]\cos 2x$