(4) The side c of a triangle ABC is determined by measuring the sides a, b and the angle C. If the possible errors of measurement are δa, δb and δC respectively, show that

$$\delta c = \cos B \cdot \delta a + \cos A \cdot \delta b + a \sin B \cdot \delta C.$$

Ans → In any $\triangle ABC$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

or $2ab\cos C = a^2+b^2-c^2$

or $c^2 = a^2+b^2-2ab\cos C$

Differentiating

60 $$2c\delta c = 2a\delta a + 2b\delta b + 2ab\sin C\,\delta C - 2\cos C\,(a\delta b + b\delta a)$$ D.C-60

or $$c\delta c = a\delta a + b.\delta b + ab\sin C\,\delta C - a\delta b\cos C - b\delta a\cos C$$

or $$c\delta c = \delta a(a - b\cos C) + \delta b(b - a\cos C) + ab\sin C\,\delta C \quad \cdots (1)$$

we know that

$$a = b\cos C + c\cos B$$

or $$a - b\cos C = c\cos B$$

Again, $$b = a\cos C + c\cos A$$

or $$b - a\cos C = c\cos A$$

Now from equation (1)

$$c\delta c = \delta a.c\cos B + \delta b.c\cos A + ab\sin C.\delta C$$

$$\because \frac{b}{\sin B} = \frac{c}{\sin C}$$

or $$b\sin C = c\sin B$$

or $$c.\delta c = \delta a.c\cos B + \delta b.c\cos A + a.c\sin B.\delta C$$

or $$\delta c = \cos B.\delta a + \cos A.\delta b + a\sin B.\delta C$$

(#) If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$,

prove that $u_x^2 + u_y^2 + u_z^2 = 2(xu_x + yu_y + zu_z)$.

Ans, Given that

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \quad \cdots (1)$$

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61 D.C. ① partially w.r.to x

$$\frac{2x}{a^2+u} - \frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x} - \frac{y^2}{(b^2+u)^2}\frac{\partial u}{\partial x} - \frac{z^2}{(c^2+u)^2}\frac{\partial u}{\partial x} = 0$$

or $$\frac{2x}{a^2+u} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x} \quad \cdots ②$$

Similarly d.c. eqn ① partially w.r.to y and z respectively we get

$$\frac{2y}{b^2+u} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial y} \quad \cdots ③$$

$$\frac{2z}{c^2+u} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial z} \quad \cdots ④$$

Multiplying equation ②, ③ & ④ by x, y and z respectively, we get

$$\frac{2x^2}{a^2+u} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] x\frac{\partial u}{\partial x}$$

$$\frac{2y^2}{b^2+u} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] y\frac{\partial u}{\partial y}$$

$$\frac{2z^2}{c^2+u} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] z\frac{\partial u}{\partial z}$$

Adding

$$2\left(\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}\right) = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right]$$

or $2 \cdot 1 = \left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$ [using eqn (1)] --- (5)

Again squaring eqn (2), (3) & (4) and adding them we get

$$4\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]^{2} \times\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]$$

or $4 = \left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right] \times\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]$ ---- (6)

Dividing equation (6) by equation (5) we get

$$\frac{4}{2} = \frac{\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]}{\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)}$$

or $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$

or $u_x^2+u_y^2+u_z^2 = 2(xu_x+yu_y+zu_z)$

INDETERMINATE FORMS

If two functions $\phi(x)$ and $\psi(x)$ be such that $\underset{x \to a}{Lt}\ \phi(x) = 0\ [\phi(a)=0]$ and $\underset{x \to a}{Lt}\ \psi(x) = 0\ [\psi(a)=0]$ then the fraction $\frac{\phi(x)}{\psi(x)}$ is considered as indeterminate form $\frac{0}{0}$ at $x \to a$ (or $x=a$).

Thus any expression $\frac{\phi(x)}{\psi(x)}$ becomes of the form $\frac{0}{0}$ at the point $x=a$ then the value of $\frac{\phi(x)}{\psi(x)}$ at that point in indefinite i.e. this form $\frac{0}{0}$ is indeterminate.

These forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\times\infty$, $\infty-\infty$, 0^0, 1^∞, ∞^0 etc are termed as indeterminate forms.

$\frac{0}{0}$ is considered as fundamental indeterminate form. For every problem corresponding to any other simple indeterminate form may be reduced to a problem corresponding to the form $\frac{0}{0}$.

This form $\frac{0}{0}$ is calculated by using L'Hospital rule. L'Hospital (1661-1704) was a french mathematician who established that

$$\underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{x \to a}{Lt} \frac{\phi^n(x)}{\psi^n(x)}$$

64 ✱ Prove that

$$\underset{x \to 0}{Lt} \frac{\sqrt{1+x}-1}{\sqrt{x}} = 0$$

D.C.64

Ans, $L.H.S = \underset{x \to 0}{Lt} \frac{\sqrt{1+x}-1}{\sqrt{x}}$ $\left[\text{Put } x=0 \text{ form } \frac{0}{0}\right]$

Applying L'Hospital's rule

$$= \underset{x \to 0}{Lt} \frac{\frac{d}{dx}\left[\sqrt{1+x}-1\right]}{\frac{d}{dx}(\sqrt{x})}$$

$$= \underset{x \to 0}{Lt} \frac{\frac{1}{2\sqrt{1+x}} - 0}{\frac{1}{2\sqrt{x}}} = \underset{x \to 0}{Lt} \frac{\sqrt{x}}{\sqrt{1+x}}$$

$$= \frac{0}{1} = 0$$

✱ Evaluate $\underset{x \to 0}{Lt} \frac{a^x - b^x}{x}$

Ans, The given limit $= \underset{x \to 0}{Lt} \frac{a^x - b^x}{x}$ $\left[\text{form } \frac{0}{0}\right]$

$$= \underset{x \to 0}{Lt} \frac{\frac{d}{dx}(a^x - b^x)}{\frac{d}{dx}(x)}$$ [By L Hopital's rule or L Hospital's rule]

$$= \underset{x \to 0}{Lt} \frac{a^x \log_e a - b^x \log_e b}{1}$$

$$= a^0 \log a - b^0 \log b$$

$$= \log a - \log b$$

$$= \log \frac{a}{b}$$

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65. Prove that

$$\underset{x\to 0}{Lt} \frac{e^x - e^{-x}}{x} = 2$$

$$L.H.S. = \underset{x\to 0}{Lt} \frac{e^x - e^{-x}}{x} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x\to 0}{Lt} \frac{e^x + e^{-x}}{1} \quad [\text{By L-Hopital's rule}]$$

$$= \frac{e^0 + e^{-0}}{1} = \frac{1+1}{1} = 2$$

$$\underset{x\to 0}{Lt} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x} = 1$$

$$L.H.S. = \underset{x\to 0}{Lt} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x} \quad \left[\frac{0}{0}\right]$$

Applying L-Hopital's rule

$$= \underset{x\to 0}{Lt} \frac{e^x + e^{-x} - 2\frac{1}{1+x}}{x\cos x + \sin x} \quad \left[\frac{0}{0}\right]$$

Again applying L-Hopital's rule

$$= \underset{x\to 0}{Lt} \frac{e^x - e^{-x} - 2\cdot\frac{(-)1}{(1+x)^2}}{\cos x - x\sin x + \cos x}$$

$$= \underset{x\to 0}{Lt} \frac{e^x - e^{-x} + \frac{2}{(1+x)^2}}{2\cos x - x\sin x}$$

$$= \frac{1-1+2}{2\cdot 1 - 0} = \frac{2}{2} = 1$$

66 Evaluate $\underset{x\to 0}{Lt} \frac{\sin x - x\cos x}{x-\sin x}$ D.C.66

Ans, The given limit $= \underset{x\to 0}{Lt} \frac{\sin x - x\cos x}{x - \sin x} \left[\frac{0}{0}\right]$

$= \underset{x\to 0}{Lt} \frac{\cancel{\cos x} + x\sin x - \cancel{\cos x}}{1-\cos x}$ [By L-Hopital's rule]

$= \underset{x\to 0}{Lt} \frac{x\sin x}{1-\cos x} \left[\frac{0}{0}\right]$

$= \underset{x\to 0}{Lt} \frac{x\cos x + \sin x}{\sin x}$ [By L-Hopital's rule]

$= \underset{x\to 0}{Lt} \frac{-x\sin x + \cos x + \cos x}{\cos x}$ [By L-Hopital's rule]

$= \underset{x\to 0}{Lt} \frac{2\cos x - x\sin x}{\cos x}$

$= \frac{2.1 - 0}{1} = 2$

Prove that $\underset{x\to 0}{Lt} \frac{\sin x + \sinh x - 2x}{x^5} = \frac{1}{60}$

Ans, L.H.S. $= \underset{x\to 0}{Lt} \frac{\sin x + \sinh x - 2x}{x^5} \left[\frac{0}{0}\right]$

Applying L Hopital's rule

$= \underset{x\to 0}{Lt} \frac{\cos x + \cosh x - 2}{5x^4} \left[\frac{0}{0}\right]$

Applying L Hopital's rule

$= \underset{x\to 0}{Lt} \frac{-\sin x + \sinh x}{20x^3} \left[\frac{0}{0}\right]$

Applying L Hopital's rule

$= \underset{x\to 0}{Lt} \frac{-\cos x + \cosh x}{60x^2} \left[\frac{0}{0}\right]$

Applying L Hopital's rule

$$= \underset{x\to 0}{Lt} \frac{\sin x + \sinh x}{120x} \quad \left[\frac{0}{0}\right]$$

Applying L Hopital's rule

$$= \underset{x\to 0}{Lt} \frac{\cos x + \cosh x}{120}$$

$$= \frac{1+1}{120} = \frac{2}{120} = \frac{1}{60}$$

Q. Evaluate $\underset{x\to 0}{Lt} \frac{\log x}{\cot x}$

Ans, The given limit

$$= \underset{x\to 0}{Lt} \frac{\log x}{\cot x} \quad \left[\frac{\infty}{\infty}\right]$$

Applying L Hopital's rule

$$= \underset{x\to 0}{Lt} \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x} = \underset{x\to 0}{Lt} \frac{(-)\sin^2 x}{x}$$

Applying L Hopital's rule

$$= \underset{x\to 0}{Lt} \frac{(-)\, 2\sin x \cos x}{1}$$

$$= (-)\, 2 \cdot \sin 0 \cdot \cos 0$$

$$= 0$$

Q. Evaluate $\underset{x\to 0}{Lt}\, x \log x$

Ans, The given limit $= \underset{x\to 0}{Lt}\, x \log x = \underset{x\to 0}{Lt} \frac{\log x}{\frac{1}{x}}$ $[0 \times \infty]$

Applying L Hopital's rule

$$= \underset{x\to 0}{Lt} \frac{\frac{1}{x}}{(-)\frac{1}{x^2}} = \underset{x\to 0}{Lt} (-) x$$

$$= 0$$

(*) Prove that

$$\underset{x\to 1}{Lt} \left(\frac{x}{x-1} - \frac{1}{\log x} \right) = \frac{1}{2}$$

L.H.S. $= \underset{x\to 1}{Lt} \left(\frac{x}{x-1} - \frac{1}{\log x} \right) \quad [\infty - \infty]$

$$= \underset{x\to 1}{Lt} \left(\frac{x \log x - x + 1}{(x-1)\log x} \right)$$

Applying L Hopital's rule

$$= \underset{x\to 1}{Lt} \frac{\cancel{x} \cdot \frac{1}{\cancel{x}} + \log x - 1}{(x-1)\cdot\frac{1}{x} + \log x}$$

$$= \underset{x\to 1}{Lt} \frac{\log x}{\frac{x-1+x\log x}{x}} = \underset{x\to 1}{Lt} \frac{x \log x}{x-1+x\log x} \left[\frac{0}{0}\right]$$

Applying L Hospital's rule

$$= \underset{x\to 1}{Lt} \frac{\cancel{x}\cdot\frac{1}{\cancel{x}} + \log x}{1-0+\cancel{x}\cdot\frac{1}{\cancel{x}} + \log x}$$

$$= \underset{x\to 1}{Lt} \frac{1+\log x}{2+\log x}$$

$$= \frac{1+\log 1}{2+\log 1} = \frac{1+0}{2+0}$$

$$= \frac{1}{2}$$