If $y = x^{n-1} \log x$

prove that $y_n = \frac{\lfloor n-1}{x}$.

Ans → Given that

$$y = x^{n-1} \log x \quad \text{-----} (1)$$

D.E. w.r. to x

$$y_1 = x^{n-1} \cdot \frac{1}{x} + \log x \cdot (n-1) x^{n-2}$$

$$\text{or } x y_1 = x^{n-1} + (n-1) \log x \cdot x \cdot x^{n-2}$$

$$= x^{n-1} + (n-1) \log x \cdot x^{n-1}$$

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$xy_1 = x^{n-1} + (n-1)\cdot y$ [from (1)] D.C. 32

D.C. $(n-1)$ times w.r.to x by Leibnitz's theorem we find

$$xy_n + {}^{n-1}C_1 \cdot 1\, y_{n-1} + 0 = \lfloor n-1 + (n-1)\, y_{n-1}$$

$$\boxed{\because D^n(x^n) = \lfloor n}$$

$$\text{or } xy_n + \cancel{(n-1)y_{n-1}} = \lfloor n-1 + \cancel{(n-1)y_{n-1}}$$

$$\text{or } xy_n = \lfloor n-1$$

$$\text{or } y_n = \frac{\lfloor n-1}{x}$$

Prove that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n \lfloor n}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right]$$

Ans, Suppose that

$$U = \frac{1}{x} \text{ and } V = \log x$$

Now, $U_n = \dfrac{(-1)^n}{x^{n+1}} \cdot \lfloor n$

$$U_{n-1} = \frac{(-1)^{n-1} \cdot \lfloor n-1}{x^n} = \frac{(-)(-1)^n \lfloor n-1}{x^n}$$

$$U_{n-2} = \frac{(-1)^{n-2} \lfloor n-2}{x^{n-1}}$$

$$= \frac{(-1)^{n} \cdot \lfloor n-2}{x^{n-1}}$$

Similarly

$$U_{n-3} = \frac{(-1)^{n-3} \lfloor n-3}{x^{n-2}} = -\frac{(-1)^{n} \lfloor n-3}{x^{n-2}}$$

...

Now

$$V_1 = \frac{1}{x}$$

$$V_2 = \frac{-1}{x^2}, \quad V_3 = \frac{2}{x^3}$$

...

$$V_n = \frac{(-1)^n \lfloor n-1}{x^n}$$

$$L.H.S = \frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{d^n}{dx^n}(UV)$$

$$= (UV)_n$$ [Now using Leibnitz's theorem]

$$= U_n V + n\, U_{n-1} V_1 + \frac{n(n-1)}{2} U_{n-2} V_2 + \dots + U V_n$$

$$= \frac{(-1)^n \lfloor n}{x^{n+1}} \cdot \log x + n \cdot \frac{(-)(-1)^n \lfloor n-1}{x^n} \cdot \frac{1}{x}$$

$$- \frac{n(n-1)}{2} \cdot \frac{(-1)^n \lfloor n-2}{x^{n-1}} \cdot \frac{1}{x^2} - \dots - \frac{1}{x} \cdot \frac{(-1)^n \lfloor n-1}{x^n}$$

$$= \frac{(-1)^n \lfloor n}{x^{n+1}} \left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n} \right]$$

$= R.H.S.$

(4) If $u_n = \frac{d^n}{dx^n}(x^n \log x)$ show that
$u_n = n\, u_{n-1} + \lfloor n-1$ and
hence deduce that
$u_n = \lfloor n \left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right)$

Ans → Given that

$$u_n = \frac{d^n}{dx^n}(x^n \log x) \quad \text{------} (1)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}(x^n \log x)\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[x^n \cdot \frac{1}{x} + n x^{n-1} \cdot \log x\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[x^{n-1} + n x^{n-1} \log x\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}(n x^{n-1} \log x) + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$$= n \cdot \frac{d^{n-1}}{dx^{n-1}}(x^{n-1} \log x) + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$$u_n = n\, u_{n-1} + \lfloor n-1 \quad \text{------} (2)$$

------proved

Putting $n = n-1$ in equ^n (2)

$$u_{n-1} = (n-1)u_{n-2} + \lfloor n-2$$

Putting this value in equ (2) we get

$$u_n = n\{(n-1)u_{n-2} + \lfloor n-2\} + \lfloor n-1$$

$$\text{or } u_n = n(n-1)u_{n-2} + n\cdot\lfloor n-2 + \lfloor n-1 \quad \ldots\ldots (3)$$

$$= \frac{n(n-1)}{\lfloor n-2}\cdot\lfloor n-2\cdot u_{n-2} + \frac{n\cdot\lfloor n-2}{(n-1)}\cdot(n-1) + \frac{n\lfloor n-1}{n}$$

$$u_n = \frac{\lfloor n}{\lfloor n-2}u_{n-2} + \frac{\lfloor n}{n-1} + \frac{\lfloor n}{n} \quad \ldots\ldots (4)$$

Putting $n = n-2$ in equ^n (2)

$$u_{n-2} = (n-2)u_{n-3} + \lfloor n-3$$

Putting this value in equ^n (3) we get

$$u_n = n(n-1)\{(n-2)u_{n-3} + \lfloor n-3\} + n\cdot\lfloor n-2 + \lfloor n-1$$

$$= n(n-1)(n-2)u_{n-3} + n(n-1)\lfloor n-3 + n\cdot\lfloor n-2 + \lfloor n-1$$

$$= \frac{n(n-1)(n-2)\lfloor n-3}{\lfloor n-3}u_{n-3} + \frac{n(n-1)(n-2)\lfloor n-3}{(n-2)} + \frac{n(n-1)\lfloor n-2}{(n-1)} + \frac{n\cdot\lfloor n-1}{n}$$

$$u_n = \frac{\lfloor n}{\lfloor n-3}u_{n-3} + \frac{\lfloor n}{(n-2)} + \frac{\lfloor n}{(n-1)} + \frac{\lfloor n}{n} \quad \ldots\ldots (5)$$

$$u_n = \frac{L n}{L 1} u_1 + \frac{Ln}{2} + \frac{Ln}{3} + \cdots + \frac{Ln}{(n-1)} + \frac{Ln}{n} \quad \cdots (6)$$

Putting $n = 1$ in equation (1)
we get

$$u_1 = \frac{d}{dx}(x \log x)$$
$$= \log x \cdot (1) + x \cdot \frac{1}{x}$$
$$u_1 = \log x + 1$$

Put the value of u_1 in eqn (6)

$$u_n = Ln(\log x + 1) + \frac{Ln}{2} + \frac{Ln}{3} + \cdots + \frac{Ln}{n}$$

$$= Ln\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$ proved

(8) If $y = (x^2-1)^n$
prove that
$$(x^2-1)y_{n+2} + 2x\, y_{n+1} - n(n+1)y_n = 0$$

Ans, Given that
$$y = (x^2-1)^n$$
D.c.w. r to x
$$y_1 = n(x^2-1)^{n-1} \cdot 2x$$
or $y_1 \cdot (x^2-1) = n \cdot (x^2-1)^n \cdot 2x$

37 or $y_1 \cdot (x^2-1) = 2nxy$ D.C.37

D.C. w.r. to x

$y_2(x^2-1) + 2xy_1 = 2nxy_1 + 2ny$

or $y_2(x^2-1) + 2y_1x(1-n) - 2ny = 0$

D.C. n times w.r. to x using Leibnitz's theorem we get

$y_{n+2}(x^2-1) + {}^nc_1\, y_{n+1}(2x) + {}^nc_2\, y_n(2) + y_{n+1}\, 2x(1-n)$
$+ {}^nc_1\, y_n \cdot 2(1-n) - 2n\, y_n = 0$

or $y_{n+2}(x^2-1) + 2nx\, y_{n+1} + \frac{n(n-1)}{\cancel{2}} y_n \cdot \cancel{2}$
$+ 2x\, y_{n+1}(1-n) + 2n \cdot (1-n)\, y_n - 2n\, y_n = 0$

or $y_{n+2}(x^2-1) + 2x \cdot y_{n+1}[\cancel{n}+1-\cancel{n}] + y_n[n^2 - n + \cancel{2n} - 2n^2 - \cancel{2n}] = 0$

or $y_{n+2}(x^2-1) + 2x\, y_{n+1} - y_n[-n-n^2] = 0$

or $y_{n+2}(x^2-1) + 2x \cdot y_{n+1} - n(n+1)\, y_n = 0$

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(4) If $y = x^2 e^x$ Show that

$$\frac{d^n y}{dx^n} = \frac{1}{2} n(n-1)\frac{d^2 y}{dx^2} - n(n-2)\frac{dy}{dx} + \frac{1}{2}(n-1)(n-2)y$$

Ans, Given that $y = x^2 e^x$ ...... (1)

D.C. w.r.t. x

$y_1 = 2xe^x + x^2 e^x$ ..... (2)

$y_2 = 2e^x + 2xe^x + 2xe^x + x^2 e^x$

$y_2 = 2e^x + 4xe^x + x^2 e^x$ ...... (3)

D.C. (1) n times w.r.t. x using Leibnitz's theorem.

$$y_n = e^x . x^2 + n\, e^x.(2x) + \frac{n(n-1)}{\cancel{2}} e^x . (\cancel{2})$$

$$y_n = x^2 e^x + 2nx\, e^x + n(n-1)\, e^x \quad ...... (4)$$

Now, R.H.S. $= \frac{1}{2} n(n-1) y_2 - n(n-2) y_1 + \frac{1}{2}(n-1)(n-2)y$

$$= \frac{1}{2} n(n-1)(2e^x + 4xe^x + x^2 e^x) - n(n-2)(2xe^x + x^2 e^x) + \frac{1}{2}(n-1)(n-2)x^2 e^x$$

$$= \frac{1}{2} n(n-1)(2 + 4x + x^2)e^x - n(n-2)(2x + x^2)e^x + \frac{1}{2}(n-1)(n-2)x^2 e^x$$

$$= x^2 e^x\left[\frac{1}{2} n(n-1) - n(n-2) + \frac{1}{2}(n-1)(n-2)\right] + 2xe^x\left[n(n-1) - n(n-2)\right] + \frac{1}{\cancel{2}} n(n-1).\cancel{2}e^x$$

$$= x^2 e^x\left[\frac{n^2 - n - 2n^2 + 4n + n^2 - 3n + 2}{2}\right] + 2xe^x\left[\cancel{n^2} - n - \cancel{n^2} + 2n\right] + n(n-1).e^x$$

$$= x^2 e^x + 2nxe^n + n(n-1)e^x$$

$$= y_n \quad [\text{from equ}^n\ (4)]$$

$$= L.H.S.$$